One-shot partial decoupling

& its application to complex quantum many-body systems

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This work was supported by CREST, JST, Grant No. JPMJCR1671.

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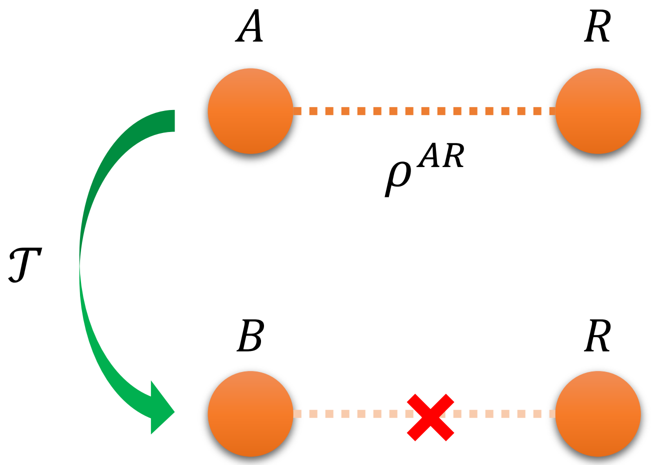
The concept of decoupling & Decoupling theorem

What is “decoupling”?



- correlated bipartite quantum state

$\mathcal{T}:\ A \rightarrow B$ (completely positive map)

- uncorrelated bipartite quantum state

$$\mathcal{T}(\rho^{AR}) \approx \tau^{B} \otimes \rho^{R}$$

Example:

$$\rho^{AR} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

$\mathcal{T}_1$: I, X with prob. 1/2

$$\mathcal{T}_1(\rho^{AR}) = \frac{1}{2} I^{B} \otimes \frac{1}{2} I^{R}$$

$$\mathcal{T}_2(\rho^{AR}) = \frac{1}{2} |+\rangle\langle +|^{B} \otimes \frac{1}{2} I^{R}$$

$\mathcal{T}_2$: projection with $|+\rangle\langle +|$

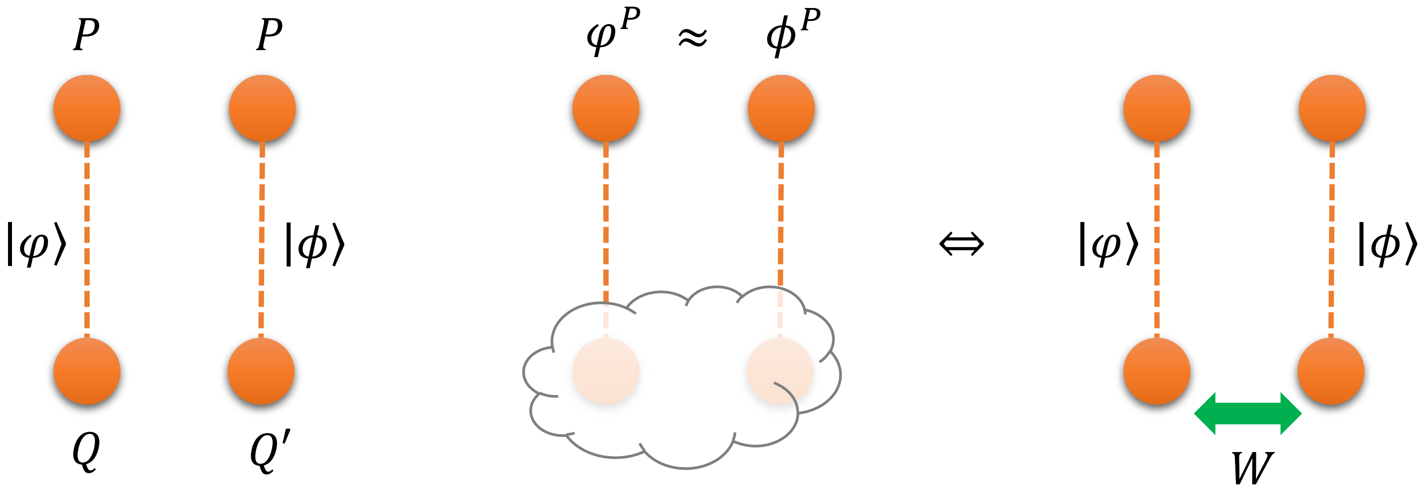
Why is decoupled state important?

- Local-isometry equivalence of purifications [Uhlamnn ‘76]

$$\mathrm{Tr}_Q\left[|\varphi\rangle\langle\varphi|^{PQ}\right] \approx \mathrm{Tr}_{Q'}\left[|\phi\rangle\langle\phi|^{PQ'}\right]$$

$$\Updownarrow$$

$$W^{Q\to Q'}|\varphi\rangle^{PQ} \approx |\phi\rangle^{PQ'} \text{ for an isometry } W.$$



Reduced states are close.

Interconvertible by an isometry.

Why is decoupled state important?

- Local-isometry equivalence of purifications [Uhlamnn ‘76]

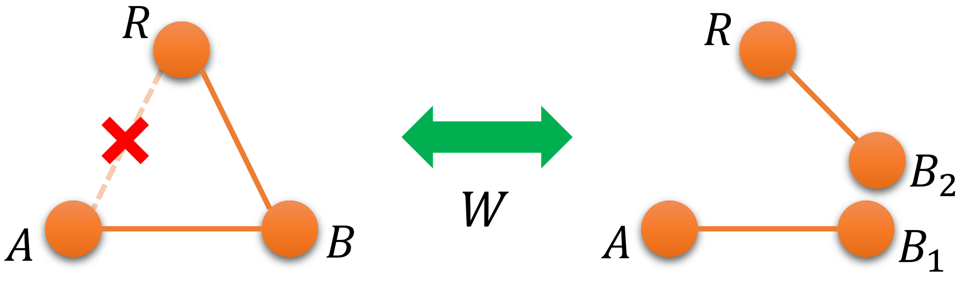
$$\mathrm{Tr}_Q\left[|\varphi\rangle\langle\varphi|^{PQ}\right] \approx \mathrm{Tr}_{Q'}\left[|\phi\rangle\langle\phi|^{PQ'}\right]$$

$$\Updownarrow$$

$$W^{Q\to Q'}|\varphi\rangle^{PQ} \approx |\phi\rangle^{PQ'} \text{ for an isometry } W.$$

- For tripartite pure state $|\Psi\rangle^{ABR}$:

$$\rho^A \otimes \sigma^R \approx \mathrm{Tr}_B[|\Psi\rangle\langle\Psi|] \Leftrightarrow W^{B\to B_1B_2}|\Psi\rangle \approx |\varphi\rangle^{AB_1}|\phi\rangle^{RB_2}$$

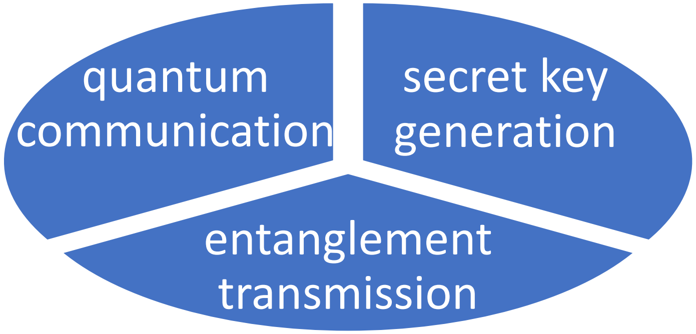


decouple A-R ⇔ pure correlation A-B

EQUIVALENT!!

Applications of decoupling

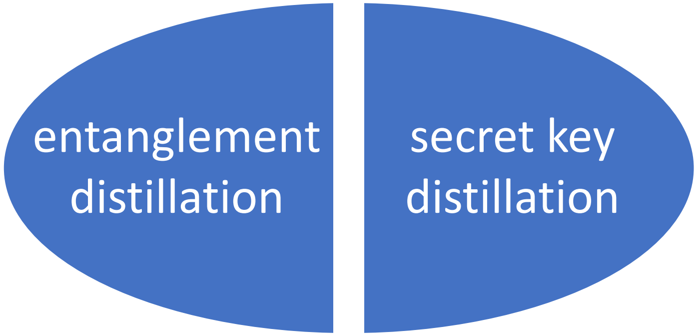
- powerful tool for analyzing q. communication tasks


quantum communication

secret key generation

entanglement transmission

(noisy quantum channel)


entanglement distillation

secret key distillation

(bipartite mixed state)

- various applications in fundamental physics

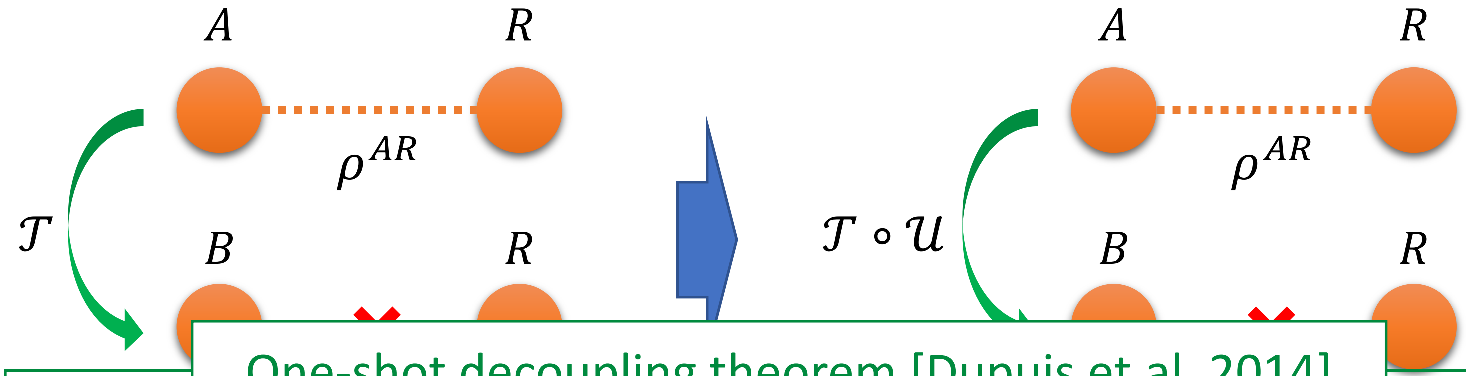
black hole information

thermalization

area law

What is "decoupling"?

Decoupling Theorem



$\mathcal{T}(\rho^{AR}) \approx \tau^B \otimes \rho^R$ $\quad$ $\mathcal{T}\circ\mathcal{U}(\rho^{AR}) \approx \tau^B \otimes \rho^R$

One-shot decoupling theorem [Dupuis et.al. 2014]

$$\mathbb{E}_U \left\| \mathcal{T}\left(U^A \rho^{AR} U^{\dagger A}\right) - \tau^B \otimes \rho^R \right\|_1 \le 2^{-\frac{1}{2}\left[\underbrace{H_{\min}(A|R)_\rho}_{\text{Correlation in } \rho^{AR}} + \underbrace{H_{\min}(A|B)_\tau}_{\text{"Decoupling power" of } \mathcal{T}}\right]}$$

Q. How precise is it (for a given ρ and $\mathcal{T}$)?

Q. How precise can we make it (for a given $\mathcal{T}$) by randomly choosing $\mathcal{U}$?

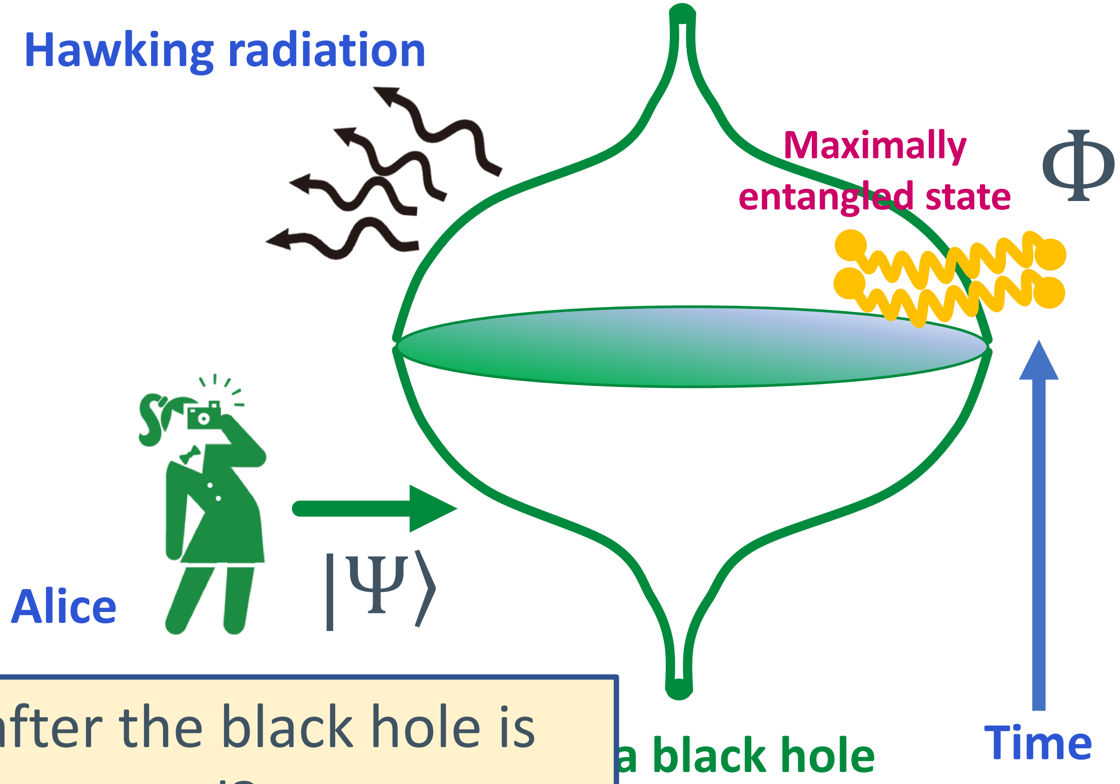
- U: Haar-random unitary on $\mathcal{H}^A$
- τ : state representation of $\mathcal{T}$ (Choi-Jamiolkowski state)
- $H_{\min}$: conditional min-entropy

➢ Corollary：There exists *a* unitary U s.t.

$$\left\| \mathcal{T}\left(U^A \rho^{AR} U^{\dagger A}\right) - \tau^B \otimes \rho^R \right\|_1 \le 2^{-\frac{1}{2}\left[H_{\min}(A|R)_\rho + H_{\min}(A|B)_\tau\right]}$$

Decoupling approach to black hole information paradox

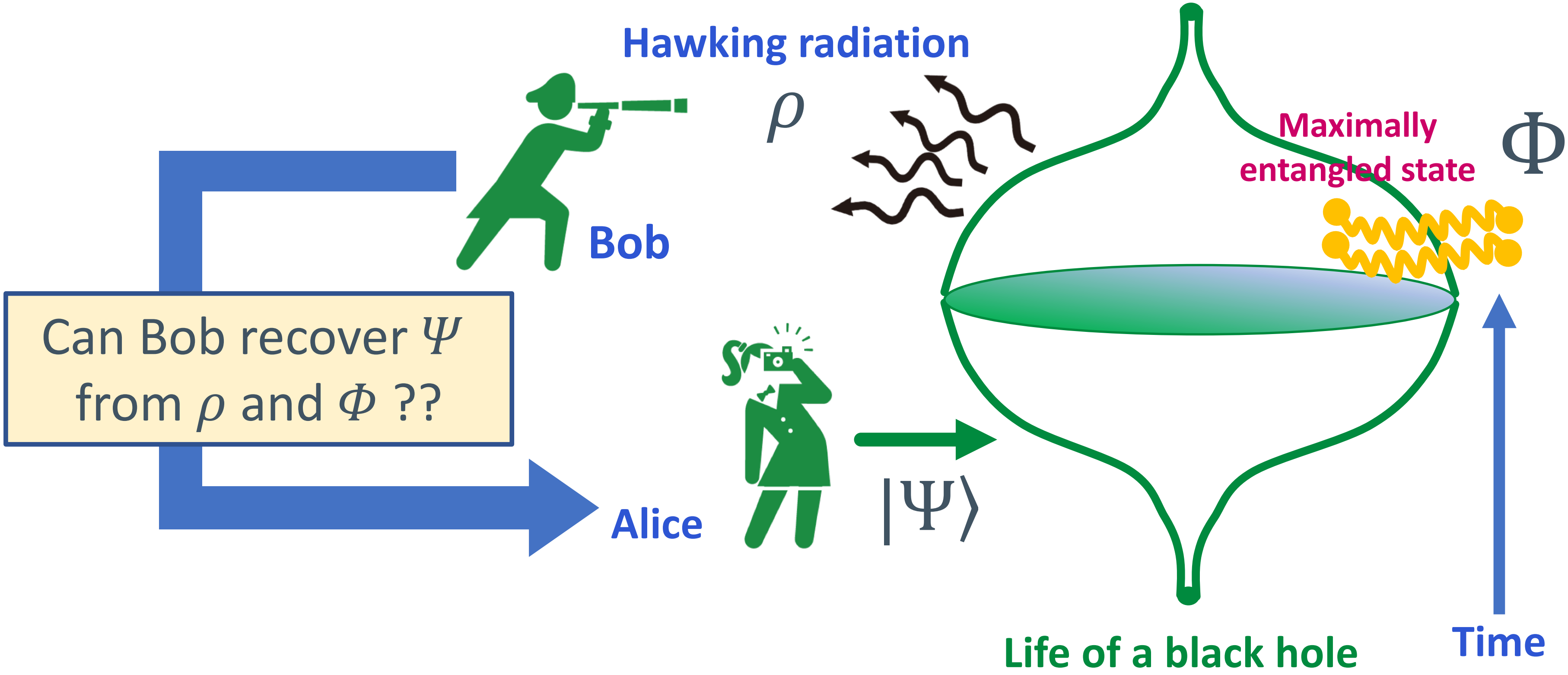
Black hole Information paradox



What happens to “$|\Psi\rangle$” after the black hole is completely evaporated?

1. Completely lost? (natural but not likely…)
2. In the Hawking radiation?

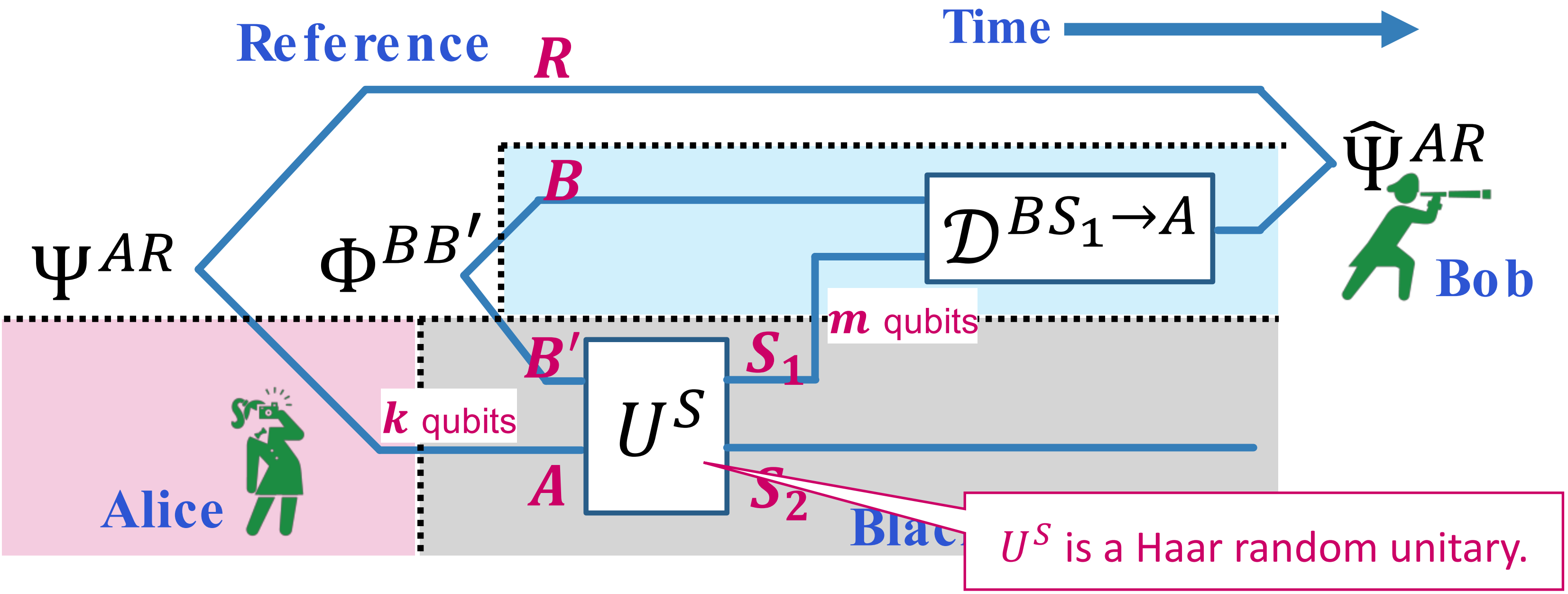
Black hole Information paradox



Can Bob recover Ψ from ρ and Φ ??

Quantum information approach by Hayden and Preskill [2007].

Hayden-Preskill's toy model (qubit-BH)

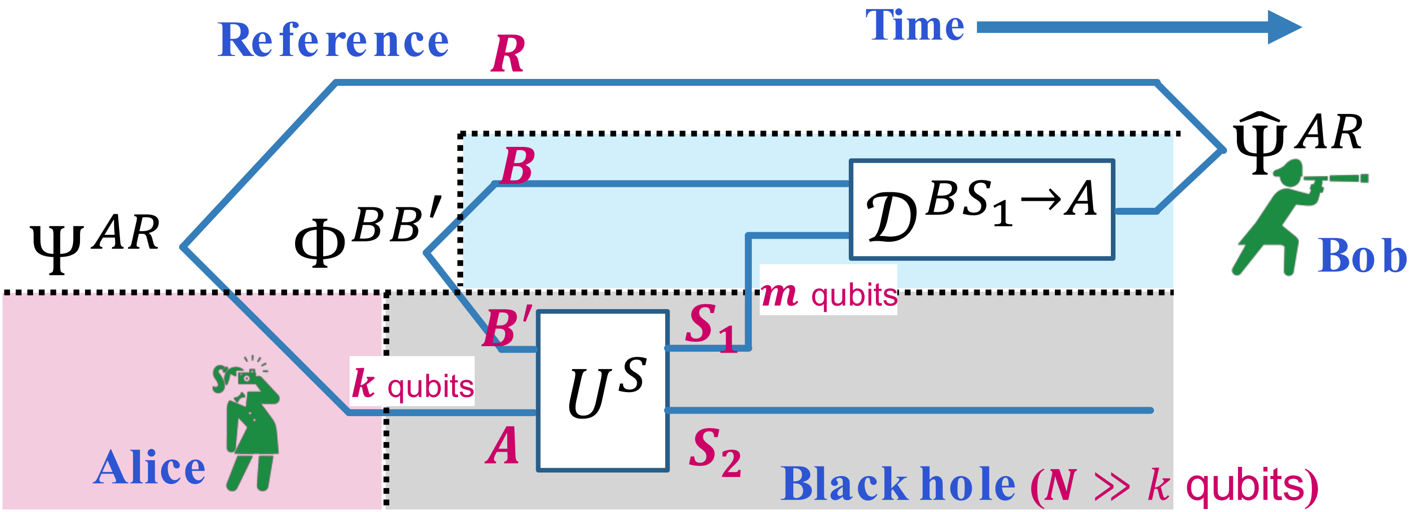


U^S is a Haar random unitary.

Question:
How large should m be for $\widehat{\Psi}^{AR}$ to be approximately Ψ^{AR}?

Assumption: U^S is a Haar random unitary.

Hayden-Preskill's toy model (qubit-BH)



HP's solution:

Assuming that the dynamics U^S is Haar random,
there exists a CPTP map $\mathcal{D}^{BS_1\to A}$, with high probability, such that

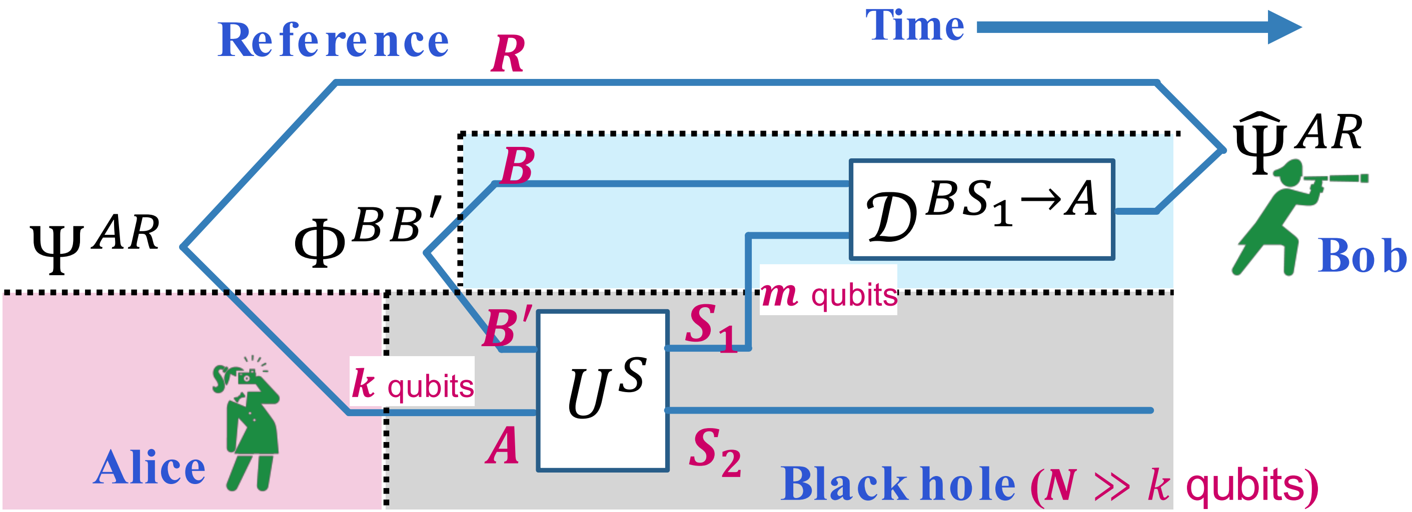
$$\left\|\widehat{\Psi}^{AR} - \Psi^{AR}\right\|_1 \approx \mathcal{O}(2^{k-m}).$$

➡ If $m \gg k$, $\widehat{\Psi}^{AR} \approx \Psi^{AR}$.

Hayden-Preskill's toy model
(qu

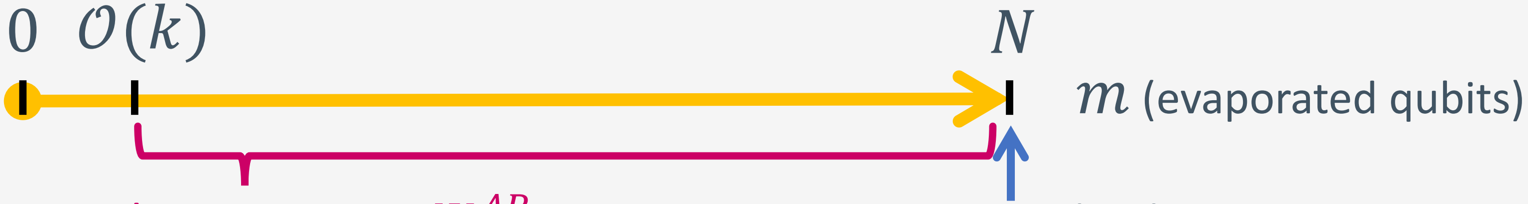
A BH is an "information mirror"…

"A black hole is hardly black at all".



HP's solution:

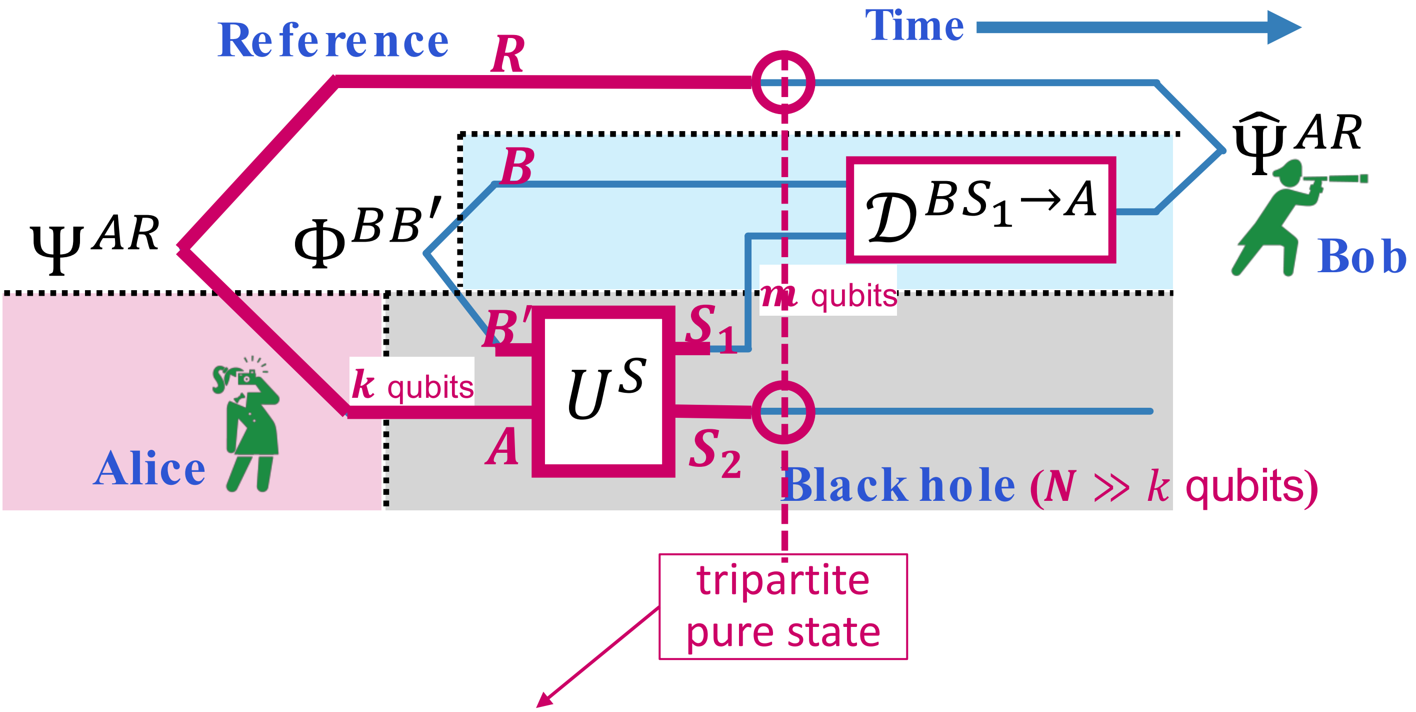
If the dynamics U^S of the BH is Haar random, with high probability,

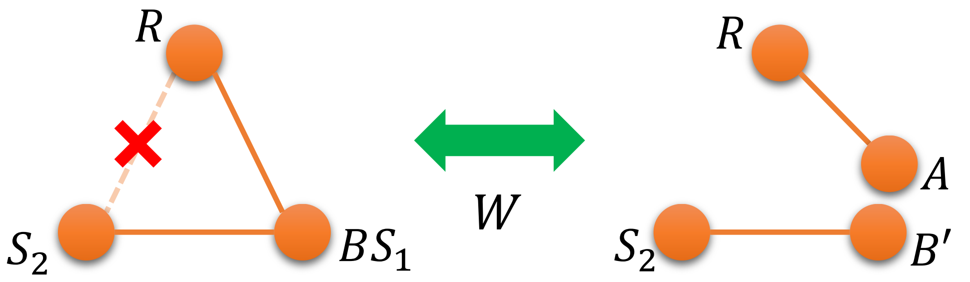


Bob can recover Ψ^{AR}.
(i.e. $\widehat{\Psi}^{AR} \approx \Psi^{AR}$)

BH is completely
evaporated.

Hayden-Preskill's approach from decoupling



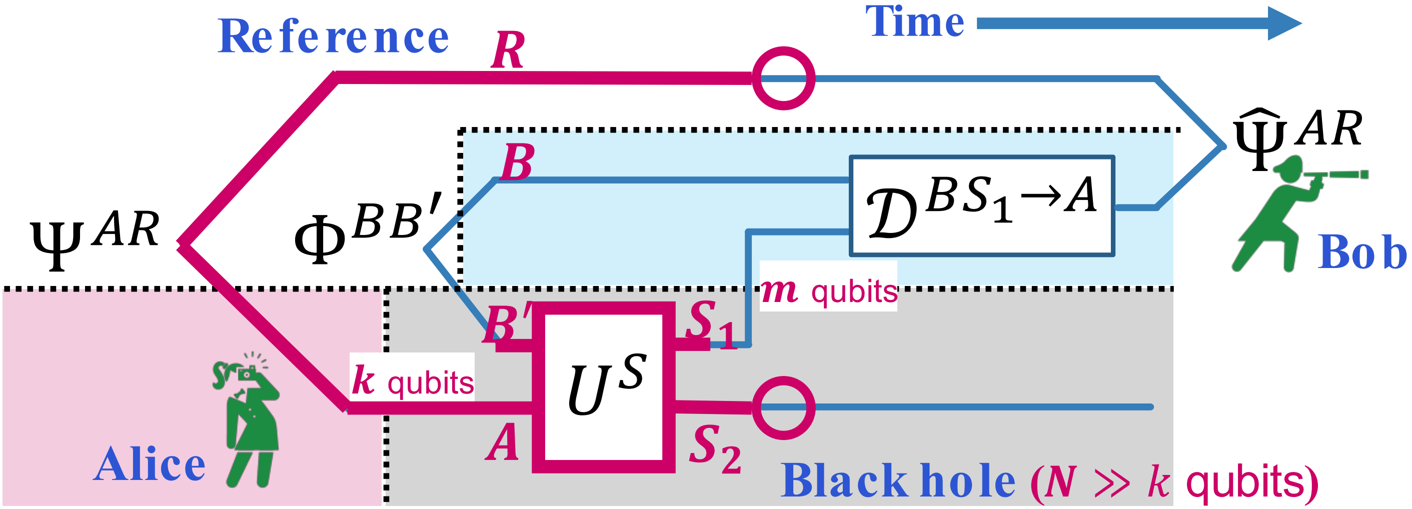


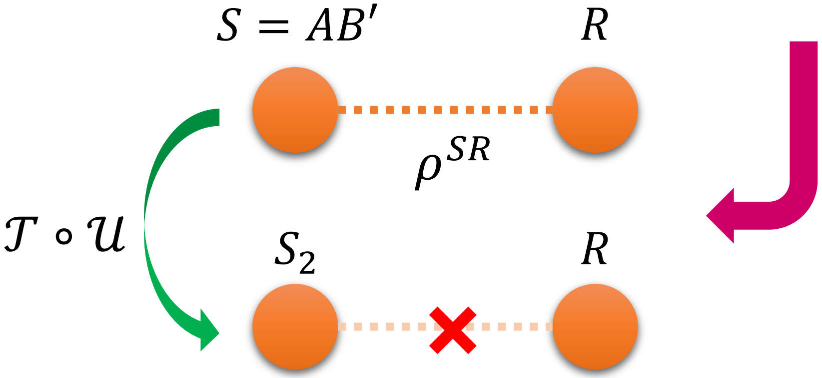
LI equivalence of purifications.

R and S_2 are decoupled.

There exists a good decoder D.

Hayden-Preskill's approach from decoupling





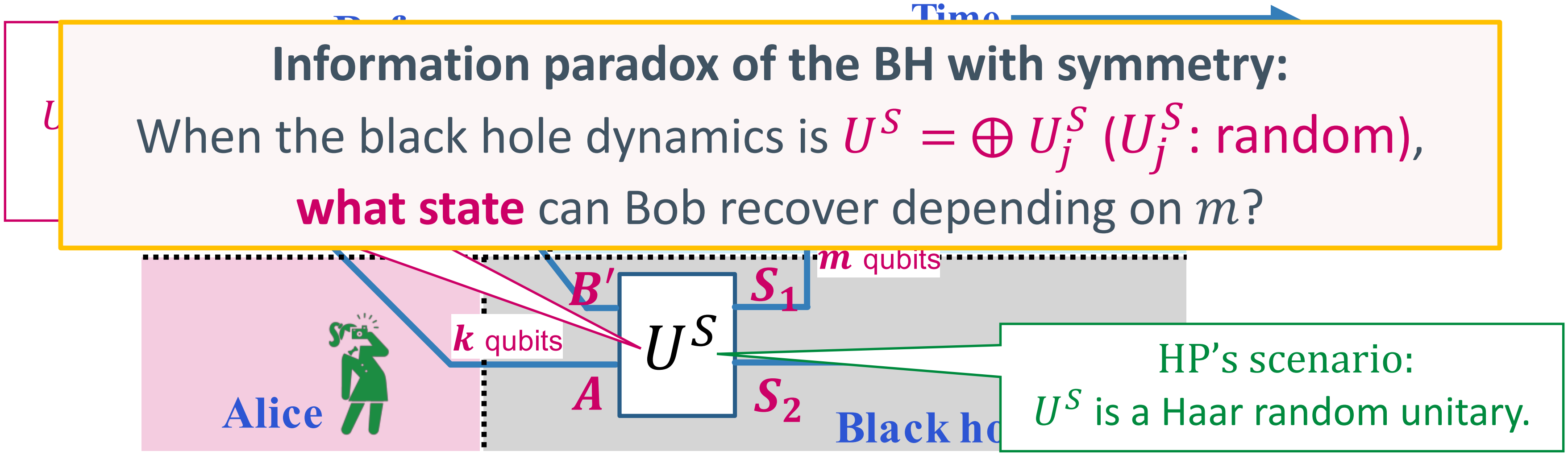
$$\mathcal{T}\circ\mathcal{U}(\rho^{SR}) \approx \tau^{S_2}\otimes\rho^{R}$$

$$\mathbb{E}_U \left\| \mathrm{Tr}_{S_1}\left(U^S \rho^{SR} U^{\dagger S}\right) - \tau^{S_2}\otimes\rho^R \right\|_1 \le 2^{-\frac{1}{2}\left[H_{\min}(S|R)_\rho + H_{\min}(S|S_2)_\tau\right]}$$

- τ : state representation of Tr_{S_1}
- $H_{\min}$: conditional min-entropy

Partial Decoupling

Hayden-Preskill's toy model (qubit-BH)



Information paradox of the BH with symmetry:
When the black hole dynamics is $U^S = \bigoplus U_j^S$ (U_j^S: random),
what state can Bob recover depending on m?

HP's scenario:
U^S is a Haar random unitary.

A HP approach was pioneering, however.....

- HP scenario is too naïve b/c U^S is assumed to be Haar random.
- What if a BH has a symmetry?
  - Conserved quantities, e.g., charges, angular momentum, spins, etc…
  - Symmetry-preserving random unitary: $U^S = \bigoplus_{j=1}^{J} (I_j^{S_l} \otimes U_j^{S_r})$.
  - For simplicity, we consider Abelian symmetries: $U^S = \bigoplus_{j=1}^{J} U_j^{S_r}$.

One-shot partial decoupling

Information paradox of the BH with symmetry:
When the black hole dynamics is $U^S = \oplus U_j^S$ (U_j^S: random),
what state can Bob recover depending on m?

HP approach in brief

1. Assume that U^S is fully random.
2. Use the one-shot decoupling theorem [Dupuis et.al. 2014].

$$U^S = \begin{pmatrix} \text{random} & \text{random} \\ \text{random} & \text{random} \end{pmatrix}$$

Our approach:

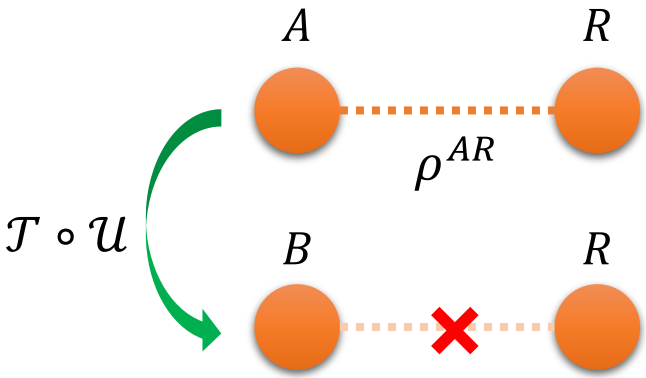
1. Assume that $U^S = \oplus U_j^S$ (U_j^S: random).
2. Prove one-shot partial decoupling theorem

- This generalization is highly non-trivial, and is of independent interest.

$$U^S = \left(\begin{array}{c:c:c} \boldsymbol{U_1^S} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & U_2^S & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \boldsymbol{U_3^S} \end{array}\right)$$

(U_j^S: random)

One-shot partial decoupling



Haar random
→ symmetry

$$\mathcal{T}\circ\mathcal{U}(\rho^{AR}) \approx \mathbb{E}_U \mathcal{T}\left(U^A \rho^{AR} U^{\dagger A}\right)$$
$$= \Sigma_{j=1}^{J} \tau_j^B \otimes \rho_j^R$$

Katrhi-Rao product $*_s$
(“block-wise” tensor product)

One-shot partial decoupling theorem [EW, Y. Nakata 2018]

$$\widetilde{\mathbb{E}}_U \left\| \mathcal{T}\left(U^A \rho^{AR} U^{\dagger A}\right) - \Sigma_{j=1}^{J} \tau_j^B \otimes \rho_j^R \right\|_1 \leq 2^{-\frac{1}{2} H_{\min}(AA'|BR)_{\tau *_s \rho}}$$

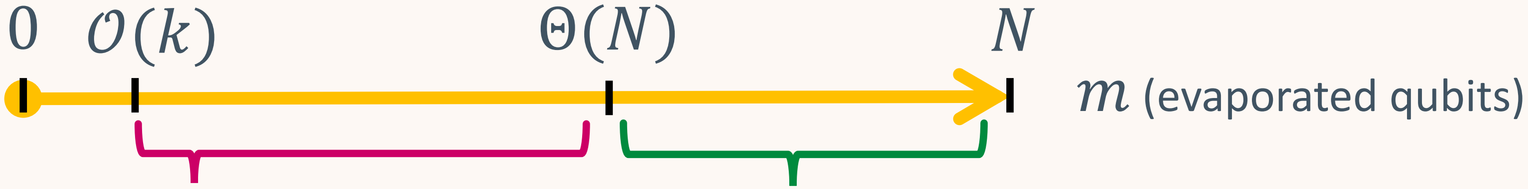
- U: Unitary of the form $U = \oplus_{j=1}^{J} U_j$ (symmetry)
- τ : state representation of $\mathcal{T}$ （Choi-Jamiolkowski state）
- $H_{\min}$: conditional min-entropy

Example: rotational symmetry

- The # of up-spins is conserved.
  - Note that it's not the $SO(3)$-symmetry (non-Abelian).

The information paradox when the BH has rotational symmetry:

Assuming that the dynamics of the BH is $U^S = U_0^S \oplus U_1^S \oplus U_2^S \oplus \cdots$



Bob can recover $\Psi^{AR}_{B.D.} (\neq \Psi^{AR})$

Bob can fully recover Ψ^{AR}.

$\Psi^{AR}_{B.D.}$

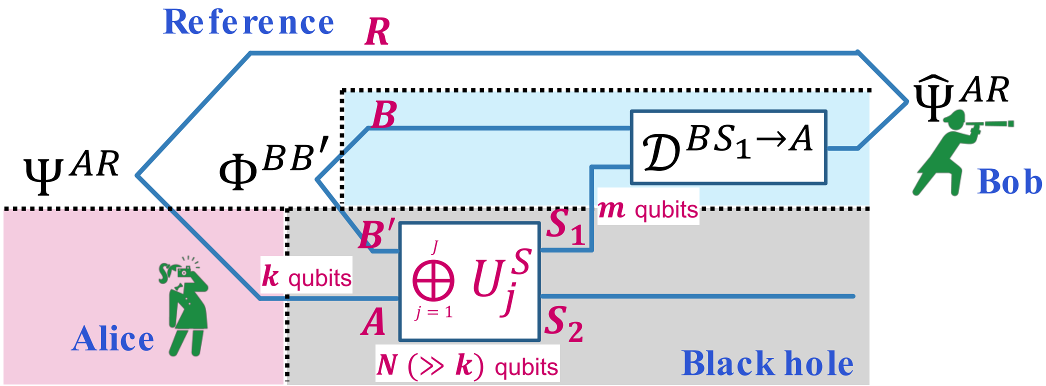
where $\gamma_{ij}(m) \in \mathbb{C}$ and $0 \leq |\gamma_{ij}| \leq 1$.

$\Psi^{AR}_{J1} \quad \Psi^{AR}_{1J} \quad \cdots \quad \Psi^{AR}_{JJ}$

~~"A black hole is hardly black at all"~~

~~The information o[illegible] the BH leaks out almost as quickly as possible.~~

Summary

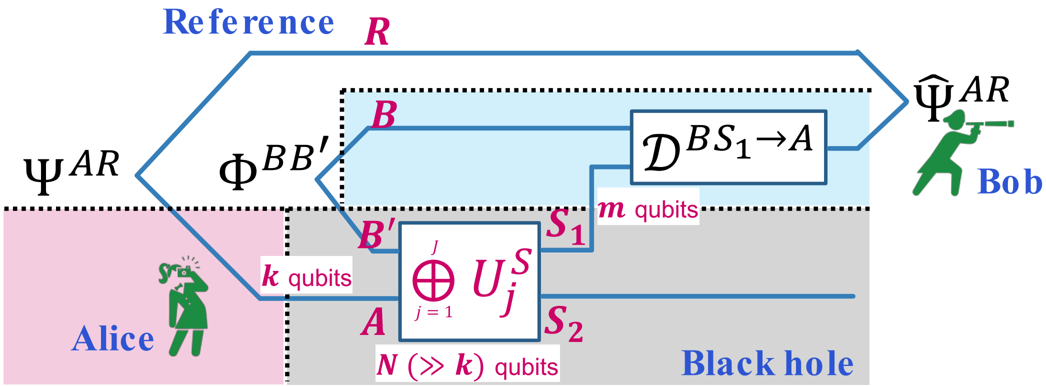


When a black hole has a symmetry,
how does the information leak out from the BH?

- Based on the one-shot partial decoupling.
- For any symmetry, all except “quantum” info. of the conserved quantity leak out quickly from the BH.
- To fully recover Ψ^{AR}, $\mathrm{O}(N)$ qubits should leak out for the rotational symmetry.

“A symmetric black hole can be black”

Future directions



When a black hole has a symmetry,
how does the information leak out from the BH?

- Non-abelian symmetries: $U^S = \bigoplus (I_j^{S_l} \otimes U_j^{S_r})$?
  - It's already done (but time was limited today….)
- Applying the one-shot partial decoupling to Q. information theory?
  - in progress…. (maybe, in next QIT)