Quantum Information & Condensed Matter Physics

New Trends in Quantum Information and Condensed Matter [Tutorial]
July 31, 2018 at ISSP, University of Tokyo

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Quantum Computing as a Reality?

Enabling Technologies

CANYON, qb quantumbenchmark, Qubitekk, QC WARE, Q-CTRL, 国盾量子 QuantumCTek, BraneCell, qasky, TUNDRASYSTEMS GLOBAL LTD., evolutionQ, Atos, SPARROW QUANTUM, SPICE LABS, RIVER LANE, STRANGE WORKS

Applications and Consulting

Optimization and ML

Q-Branch, QBITLOGIC, BOHR, CQC Cambridge Quantum Computing, Qindom Inc., ZAPATA, SolidStateAI

Chemicals, Materials, and Biotech

ProteinQure, OTI, QULAB

Cyber Security and Networks

ISARA, Quintessence Labs, KETS, id Quantique, Qunnect

Consulting

1QBit, Quantika, Q, Artiste-qb.net, Quantum Consultants

Hardware and Software

Universal Superconducting

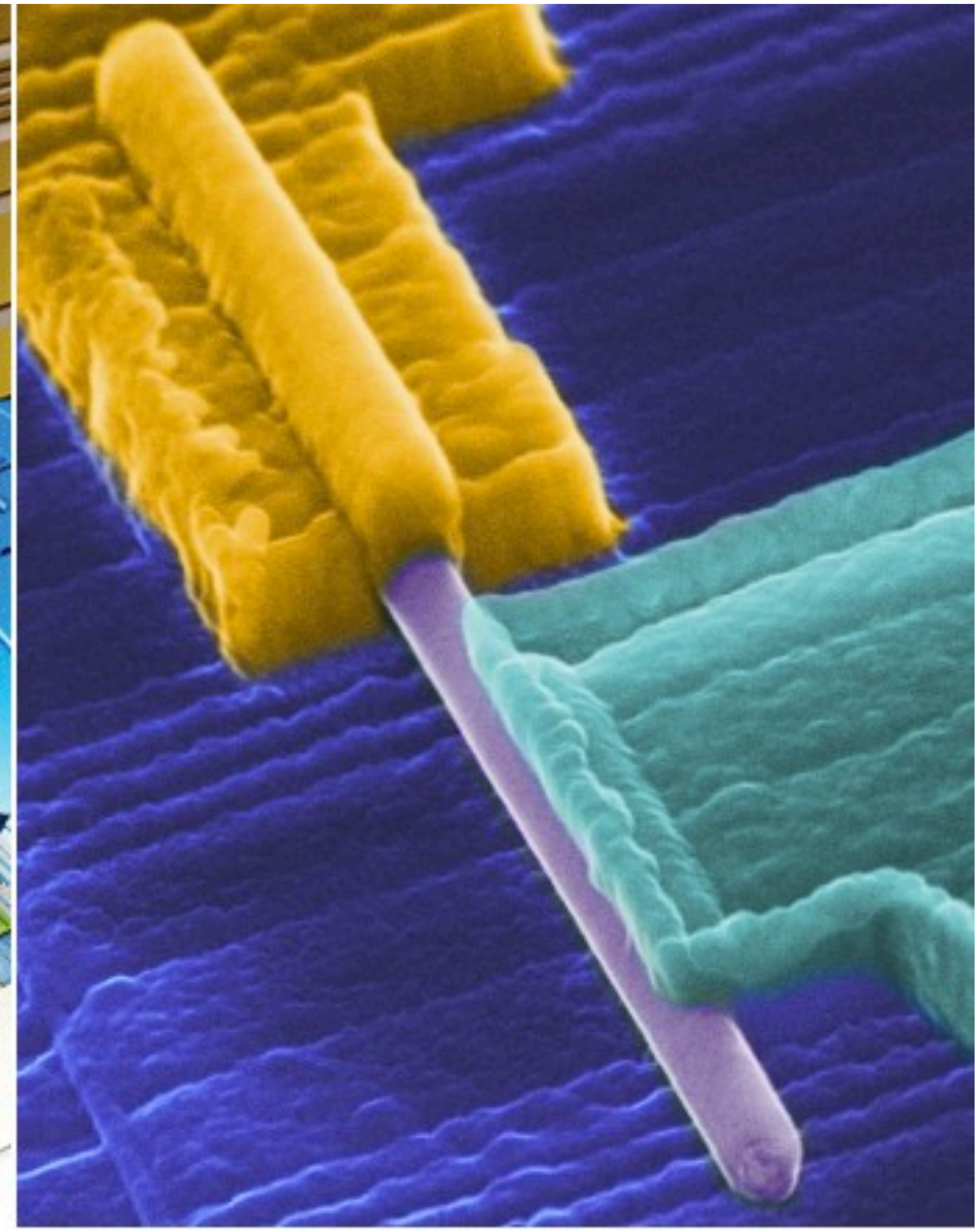
rigetti, intel, IBM, qci, Google, Alibaba Group, SeeQC

Other Universal

Microsoft, NOKIA Bell Labs, QuTech, hp, XANADU, IONQ, EeroQ, TURING

Annealing

D-Wave The Quantum Computing Company, FUJITSU, Quantum Computing Inc., QILIMANJARO



Quantum computing devices
superconducting qubit
topological qubit (Majorana fermion ....)
......

condensed-matter physics

Today's Talk

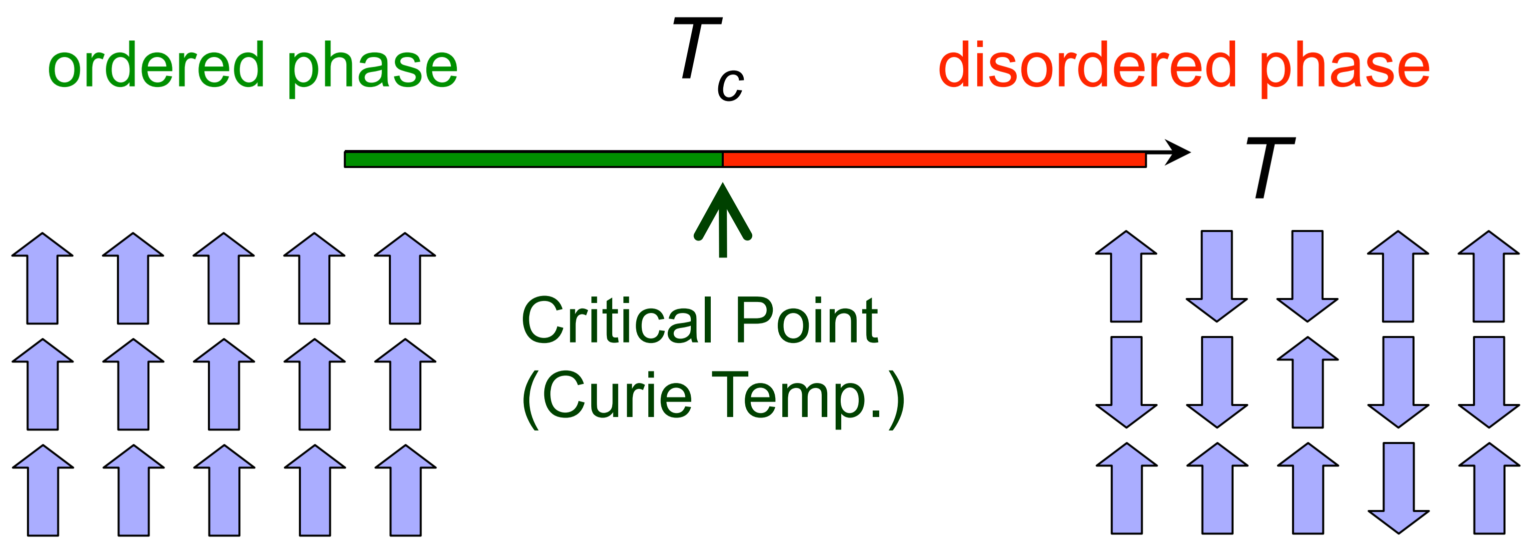
Quantum Information

Condensed Matter Physics
Statistical Mechanics

Condensed Matter Physics

- to understand physical properties of materials

- classification of states of matter into distinct "phases"



simple model: (classical) Ising model

$$\mathcal{H} = J \sum_{\langle j,k \rangle} \sigma_j^z \sigma_k^z$$

Classical Statistical Mechanics

In the equilibrium, each configuration appears with the probability

$$p_{\{\sigma_j\}} = \frac{1}{Z} e^{-\beta \mathcal{H}(\{\sigma_j\})}$$

(Gibbs ensemble)

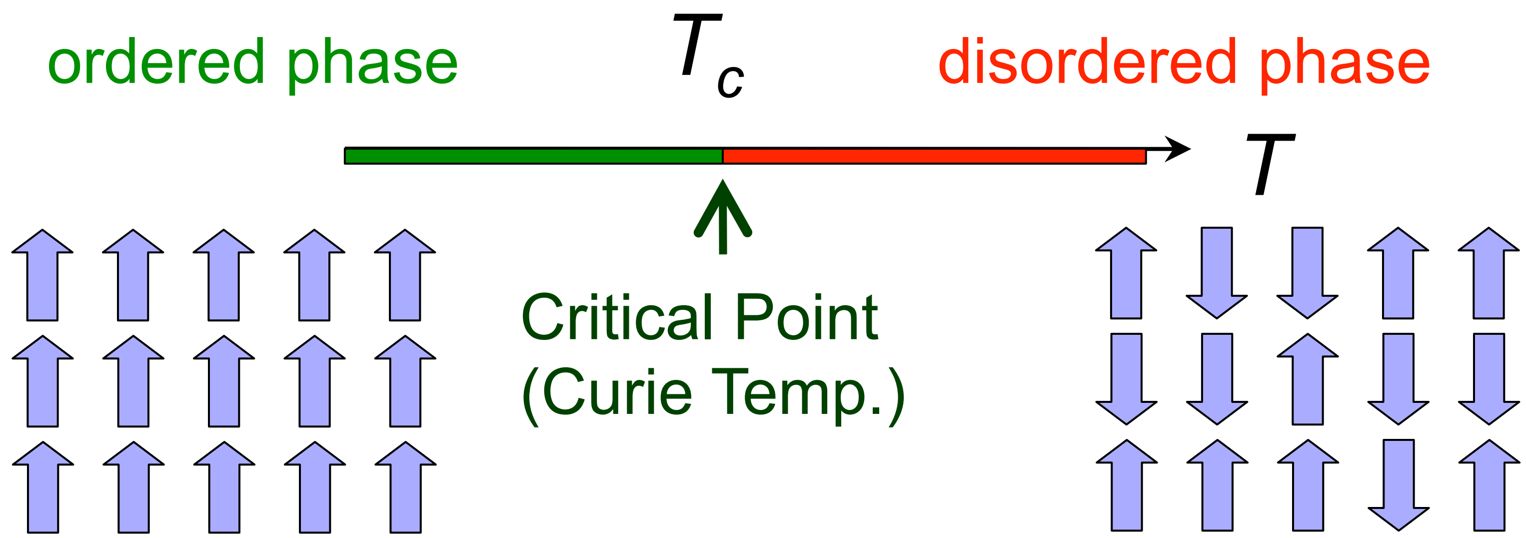
$$Z = \sum_{\{\sigma_j\}} e^{-\beta \mathcal{H}(\{\sigma_j\})}$$

$$\beta = \frac{1}{k_B T}$$

Statistical expectation value

$$\langle \mathcal{O} \rangle \equiv \frac{1}{Z} \sum_{\langle \sigma_j \rangle} \mathcal{O} e^{-\beta \mathcal{H}(\{\sigma_j\})}$$

Order Parameter



simple model: (classical) Ising model $\mathcal{H} = J \sum_{\langle j,k \rangle} \sigma_j^z \sigma_k^z$

In the case of Ising model, naively the order is characterized by the "order parameter" $m = \langle \sigma^z \rangle$

$m = 0$: disordered
$m \neq 0$: ordered

However, symmetry implies m=0 always!!

Spontaneous Symmetry Breaking

The ordered phase exhibits "spontaneous symmetry breaking" (SSB)

How to characterize the SSB?

i) infinitesimal field trick

$$\mathcal{H} = J \sum_{\langle j,k \rangle} \sigma_j^z \sigma_k^z - h \sum_j \sigma_j^z$$

$$m = \lim_{h \to +0} \lim_{V \to \infty} \langle \sigma_j \rangle$$

h: magnetic field

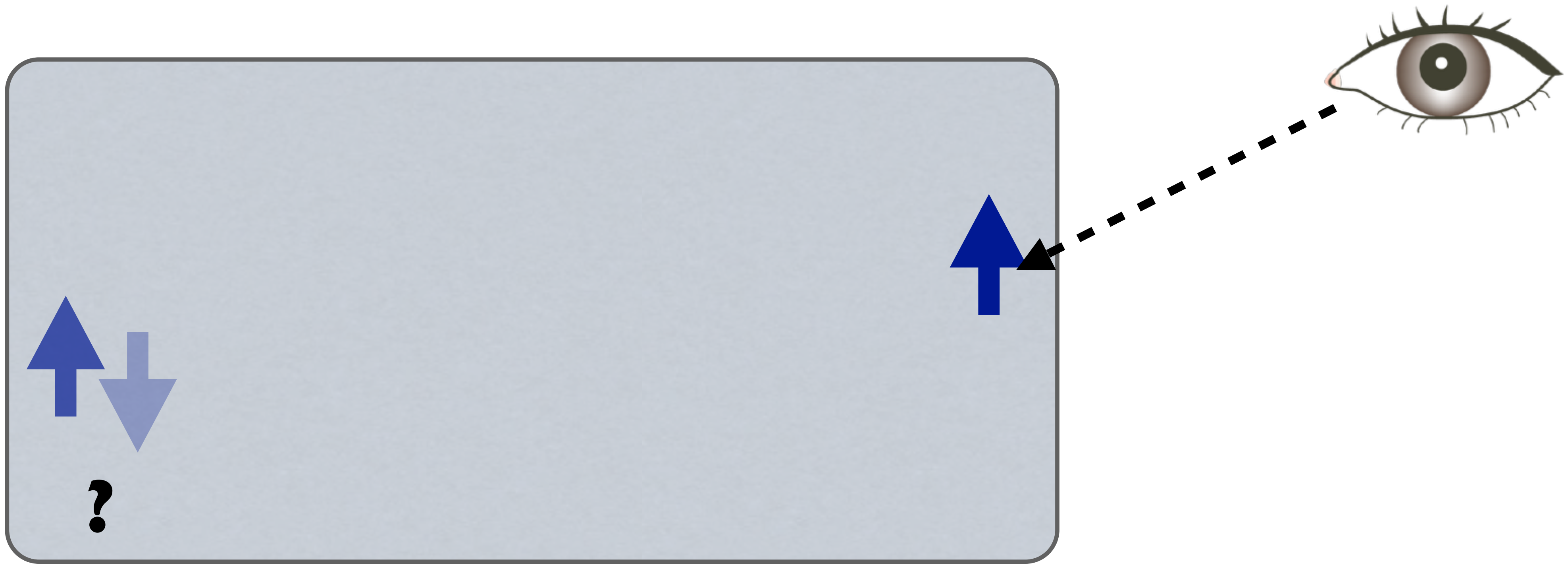
m = 0 : disordered

m ≠ 0 : ordered

ii) long-range order (correlation)

$$m^2 = \lim_{r_{jk} \to \infty} \langle \sigma_j^z \sigma_k^z \rangle - \langle \sigma_j^z \rangle \langle \sigma_k^z \rangle$$

Correlation and Information



Mutual information (classical)

$$S(X) \equiv -\sum_x p_x \log p_x$$

$$I(X,Y) \equiv S(X) + S(Y) - S(X,Y)$$

Assuming the spin inversion and exchange symmetry,

$$I(\sigma_1, \sigma_2) = \frac{1}{2}\log\left(1 - \langle\sigma_1\sigma_2\rangle^2\right) + \frac{\langle\sigma_1\sigma_2\rangle}{2}\log\left(\frac{1+\langle\sigma_1\sigma_2\rangle}{1-\langle\sigma_1\sigma_2\rangle}\right)$$

maximum mutual information $\log 2$ for $\langle\sigma_1\sigma_2\rangle = \pm 1$

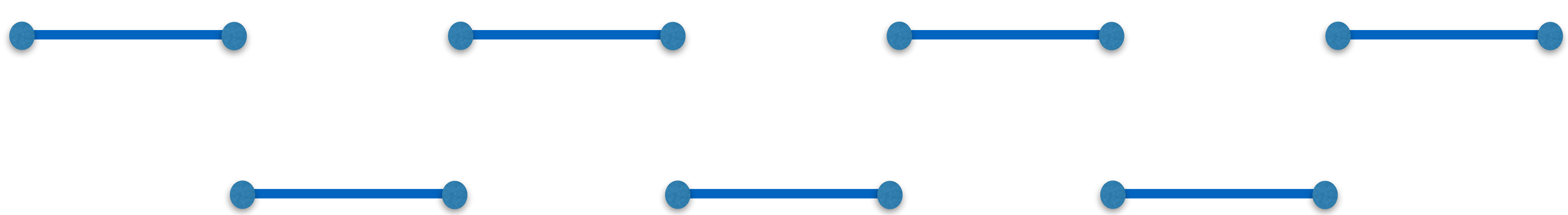
Is the single spin enough?

$$\lim_{r_{jk}\to\infty}\langle\sigma_j\sigma_k\rangle = 0$$

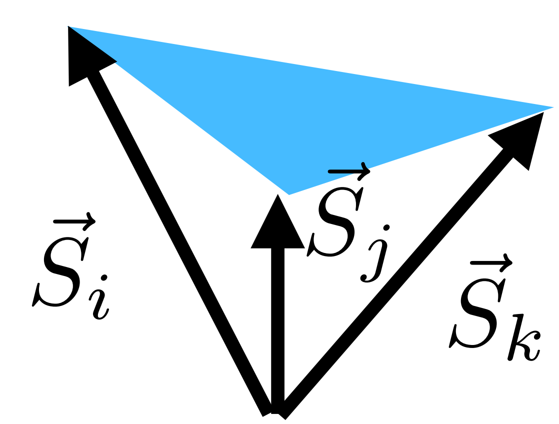
no magnetization
⇒ no long-range order?

No! There can be different types of long-range orders…

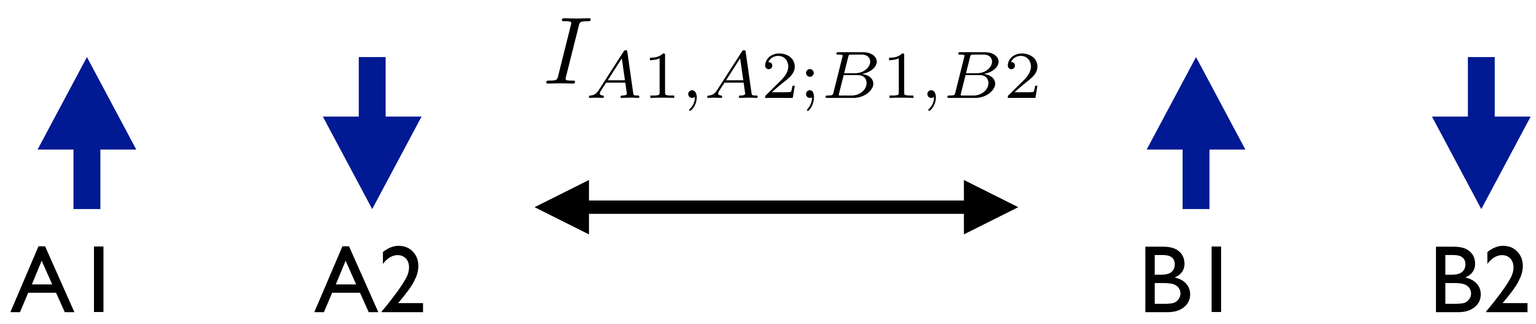
spontaneous dimerization $\qquad \vec{S}_j \cdot \vec{S}_{j+1}$



"scalar chirality" $\qquad \vec{S}_i \cdot \left(\vec{S}_j \times \vec{S}_k\right)$



Mutual Information Between Spin-Pairs

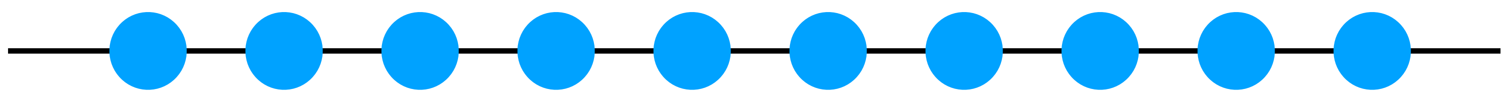


$$
\begin{aligned}
I_{A1,A2:B1,B2} =& \frac{1}{8}\{(1-2\langle\sigma_{A1}\sigma_{A2}\rangle - \langle\sigma_{A1}\sigma_{B1}\rangle + 2\langle\sigma_{A1}\sigma_{B2}\rangle - \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle) \\
&\quad \log\frac{1-2\langle\sigma_{A1}\sigma_{A2}\rangle - \langle\sigma_{A1}\sigma_{B1}\rangle + 2\langle\sigma_{A1}\sigma_{B2}\rangle - \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle}{(1-\langle\sigma_{A1}\sigma_{A2}\rangle)^2} \\
&+ 2(1-\langle\sigma_{A1}\sigma_{B1}\rangle + \langle\sigma_{A2}\sigma_{B2}\rangle - \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle)\log\frac{1-\langle\sigma_{A1}\sigma_{B1}\rangle + \langle\sigma_{A2}\sigma_{B2}\rangle - \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle}{1-\langle\sigma_{A1}\sigma_{A2}\rangle^2} \\
&+ (1+2\langle\sigma_{A1}\sigma_{A2}\rangle - \langle\sigma_{A1}\sigma_{B1}\rangle - 2\langle\sigma_{A1}\sigma_{B2}\rangle - \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle) \\
&\quad \log\frac{1+2\langle\sigma_{A1}\sigma_{A2}\rangle - \langle\sigma_{A1}\sigma_{B1}\rangle - 2\langle\sigma_{A1}\sigma_{B2}\rangle - \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle}{(1+\langle\sigma_{A1}\sigma_{A2}\rangle)^2} \\
&+ 2(1+\langle\sigma_{A1}\sigma_{B1}\rangle - \langle\sigma_{A2}\sigma_{B2}\rangle - \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle)\log\frac{1+\langle\sigma_{A1}\sigma_{B1}\rangle - \langle\sigma_{A2}\sigma_{B2}\rangle - \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle}{1-\langle\sigma_{A1}\sigma_{A2}\rangle^2} \\
&+ (1-2\langle\sigma_{A1}\sigma_{A2}\rangle + \langle\sigma_{A1}\sigma_{B1}\rangle - 2\langle\sigma_{A1}\sigma_{B2}\rangle + \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle) \\
&\quad \log\frac{1-2\langle\sigma_{A1}\sigma_{A2}\rangle + \langle\sigma_{A1}\sigma_{B1}\rangle - 2\langle\sigma_{A1}\sigma_{B2}\rangle + \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle}{(1-\langle\sigma_{A1}\sigma_{A2}\rangle)^2} \\
&+ (1+2\langle\sigma_{A1}\sigma_{A2}\rangle + \langle\sigma_{A1}\sigma_{B1}\rangle + 2\langle\sigma_{A1}\sigma_{B2}\rangle + \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle) \\
&\quad \log\frac{1-2\langle\sigma_{A1}\sigma_{A2}\rangle + \langle\sigma_{A1}\sigma_{B1}\rangle - 2\langle\sigma_{A1}\sigma_{B2}\rangle + \langle\sigma_{A2}\sigma_{B2}\rangle + \langle\sigma_{A1}\sigma_{A2}\sigma_{B2}\sigma_{B1}\rangle}{(1-\langle\sigma_{A1}\sigma_{A2}\rangle)^2}\}
\end{aligned}
$$

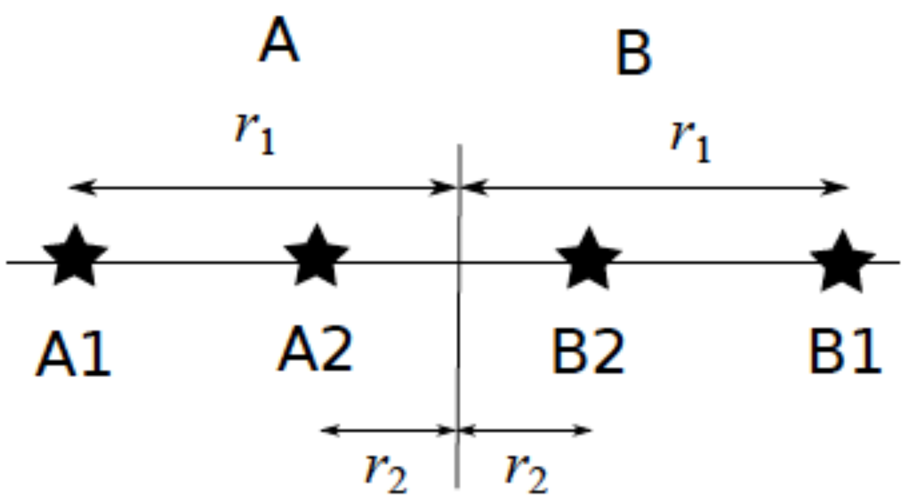
Equilibrium can be Simple

1D Ising model

$$\mathcal{H} = -J \sum_j \sigma_j \sigma_{j+1}$$



No SSB at any *T*>0, but exactly solvable



$$\langle \sigma_{A1} \sigma_{B1} \rangle = (\tanh \beta)^{2r_1}$$

$$\langle \sigma_{A1} \sigma_{A2} \sigma_{B2} \sigma_{B1} \rangle = \tanh^{2(r_1 - r_2)} \beta$$

$$I_{A1,A2;B1,B2} = I_{A2;B2}$$

A1 and B1 can "talk to each other"
only through A2 and B2!
(cf. transfer matrix solution)

Quantum Statistical Mechanics

$\mathcal{H}$ Hamiltonian now a Hermitean operator

huge Hilbert space for a macroscopic system:
dimension = 2^N for N spin-1/2

$$\rho_{\rm eq} = \frac{1}{Z} e^{-\beta\mathcal{H}} \qquad\qquad Z = {\rm Tr} e^{-\beta\mathcal{H}}$$

$$\langle \mathcal{O} \rangle = {\rm Tr}\left(\mathcal{O}\rho_{\rm eq}\right)$$

Reduces to classical statistical mechanics
when Hamiltonian is diagonal (in a local basis)

$T \to 0$ ($\beta \to \infty$)

$$\langle \mathcal{O} \rangle = \langle \Psi_0 | \mathcal{O} | \Psi_0 \rangle$$

$|\Psi_0\rangle$ ground state

Quantum Fluctuations

Example: quantum transverse Ising model

$$\mathcal{H} = -\sum_{\langle j,k \rangle} \sigma_j^z \sigma_k^z - \Gamma \sum_j \sigma_j^x$$

ordering

favors spins aligned to the same direction $\uparrow\uparrow$ or $\downarrow\downarrow$

disordering

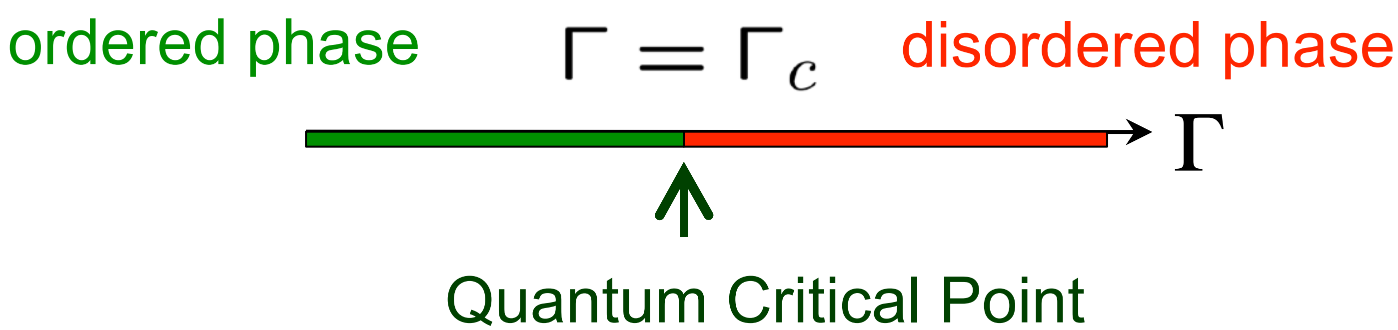
flips the spin $\uparrow \leftrightarrow \downarrow$

As a result, even the ground state is nontrivial!

Quantum Phase Transition

in the QTI model

$$\mathcal{H} = -\sum_{\langle j,k \rangle} \sigma_j^z \sigma_k^z - \Gamma \sum_j \sigma_j^x$$



Very similar to the thermal phase transition in the classical Ising model
In fact, there is a mathematical mapping between
quantum transverse Ising model in *d* space dimensions
and classical transverse Ising model in *d*+1 space dimensions

SSB and Order Parameter

Can be defined similarly to the classical case

i) infinitesimal field trick

$$m = \lim_{h \to +0} \lim_{V \to \infty} \langle \sigma_j \rangle$$

m = 0 : disordered
m ≠ 0 : ordered

ii) long-range order (correlation)

$$m^2 = \lim_{r_{jk} \to \infty} \langle \sigma_j^z \sigma_k^z \rangle - \langle \sigma_j^z \rangle \langle \sigma_k^z \rangle$$

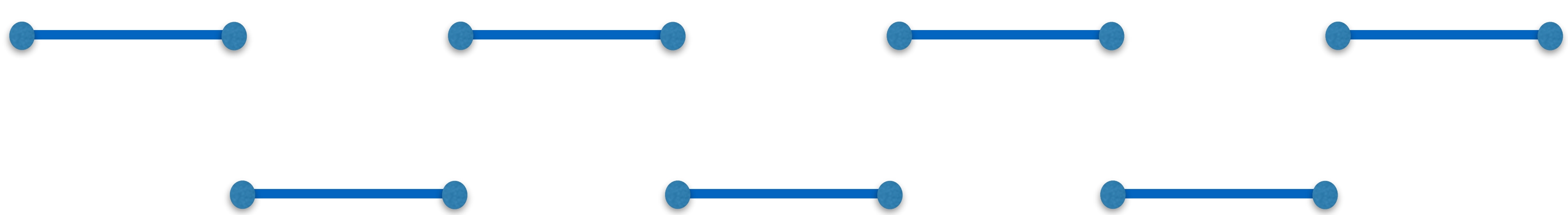
Is the single spin enough?

$$\lim_{r_{jk}\to\infty} \langle \sigma_j \sigma_k \rangle = 0$$

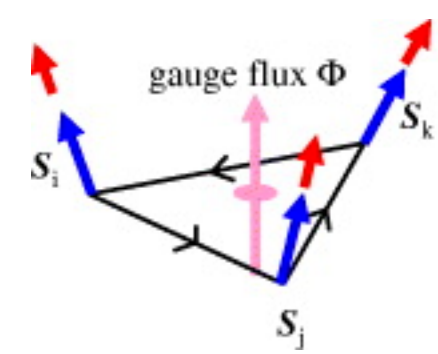
no magnetization
$\Rightarrow$ no long-range order?

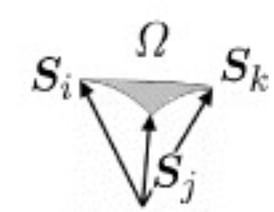
No! There can be different types of long-range orders…

spontaneous dimerization $\vec{S}_j \cdot \vec{S}_{j+1}$



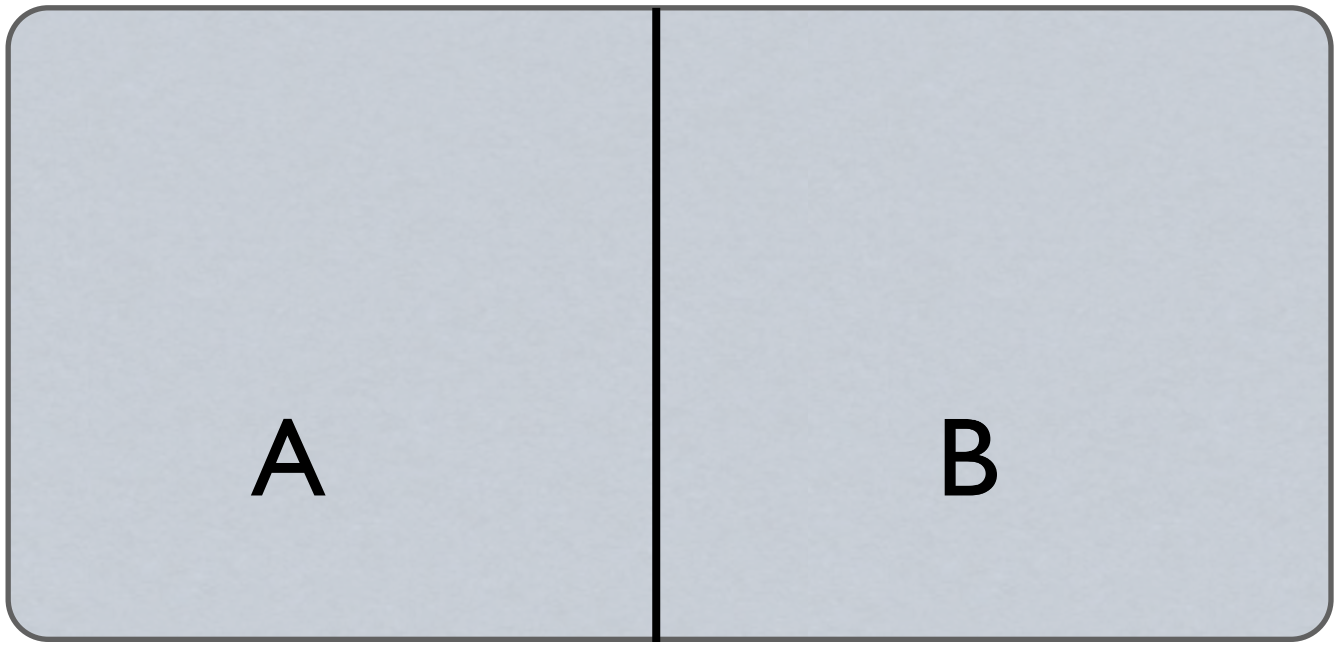
"scalar chirality" $\vec{S}_i \cdot \left(\vec{S}_j \times \vec{S}_k\right)$





Also the case in quantum statistical mechanics
(and even more subtleties…)

Characterizing Arbitrary Order?



Suppose that the total system is in a pure state
(e.g. the ground state)

No entanglement between A & B → no correlation

$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$

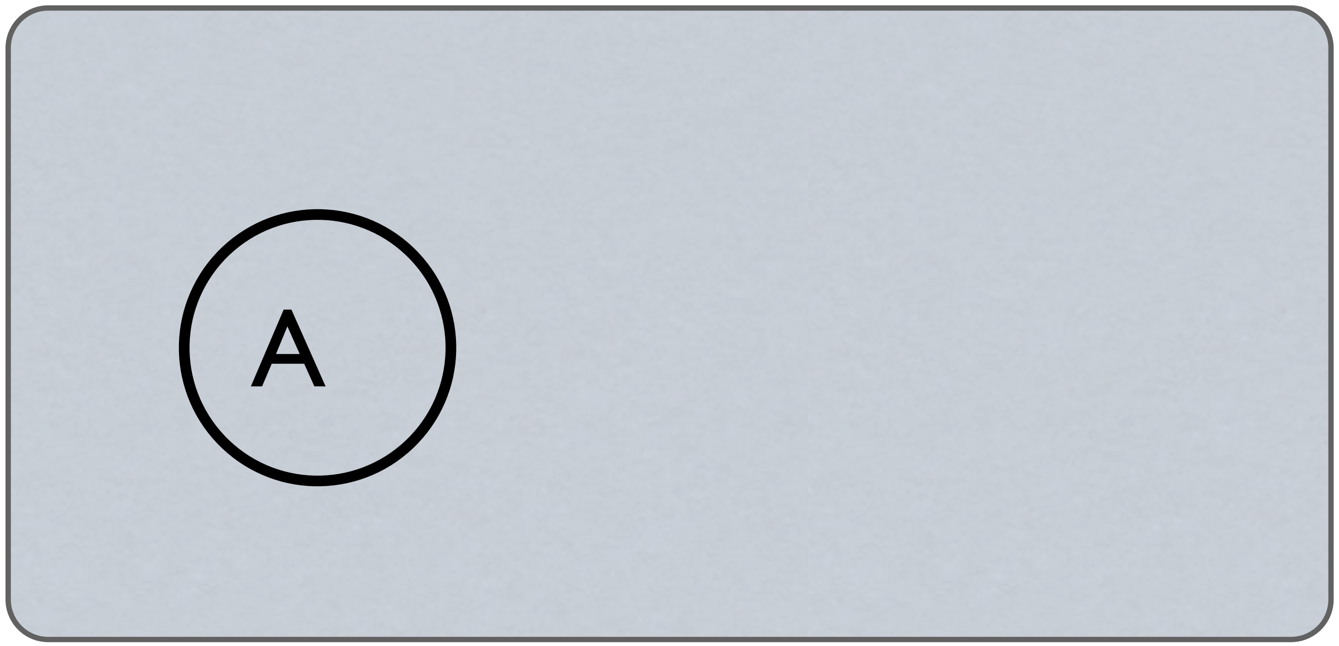
measure of entanglement

$$S_E = -\mathrm{Tr}\left[\rho_A \log \rho_A\right]$$

(von Neumann) entanglement entropy

Scaling of Entanglement

Consider an infinite system



Entanglement entropy between A & outside

Typical (random) state: ρ_A “random”

N_A eigenvalues $\sim 1/N_A$ N_A : number of spins in A

$S_E \propto N_A \propto \mathrm{Vol}_A$ “volume law”

Foundation of Statistical Mechanics

Gibbs ensemble: $$\rho_{\mathrm{eq}} = \frac{1}{Z} e^{-\beta\mathcal{H}}$$

Why? (foundation of statistical mechanics)

Modern view: the "true" density matrix is not necessarily given by the Gibbs ensemble

the entire system may be in a pure state

Why does statistical mechanics work, then?

Physical observables are (mostly) local

If the reduced density matrix for any local region is identical to that from the Gibbs ensemble, such a state is "thermal"

(indistinguishable from the Gibbs ensemble)

Typical Pure State is Thermal

Consider an energy shell $$\mathcal{H}|\Psi_\alpha\rangle = E_\alpha|\Psi_\alpha\rangle$$

$$E < E_\alpha < E + \Delta E$$

Hilbert space of the shell (superposition of the eigenstates)

Typical pure state in the Hilbert space
(with respect to the Haar measure) $$|\Psi\rangle = \sum_\alpha c_\alpha |\Psi_\alpha\rangle$$

$$\rho_A \sim \mathrm{Tr}_{\bar{A}}\left(\rho^{\mathrm{mc}}\right) \sim \mathrm{Tr}_{\bar{A}}\left(\rho_{\mathrm{eq}}\right)$$

Pospescu (2006), Sugita (2006), Tasaki (2015), ...

Thermalization from an Initial State

Generic state
(not necessarily typical)
within the energy shell

$$|\Psi\rangle = \sum_\alpha c_\alpha |\Psi_\alpha\rangle$$

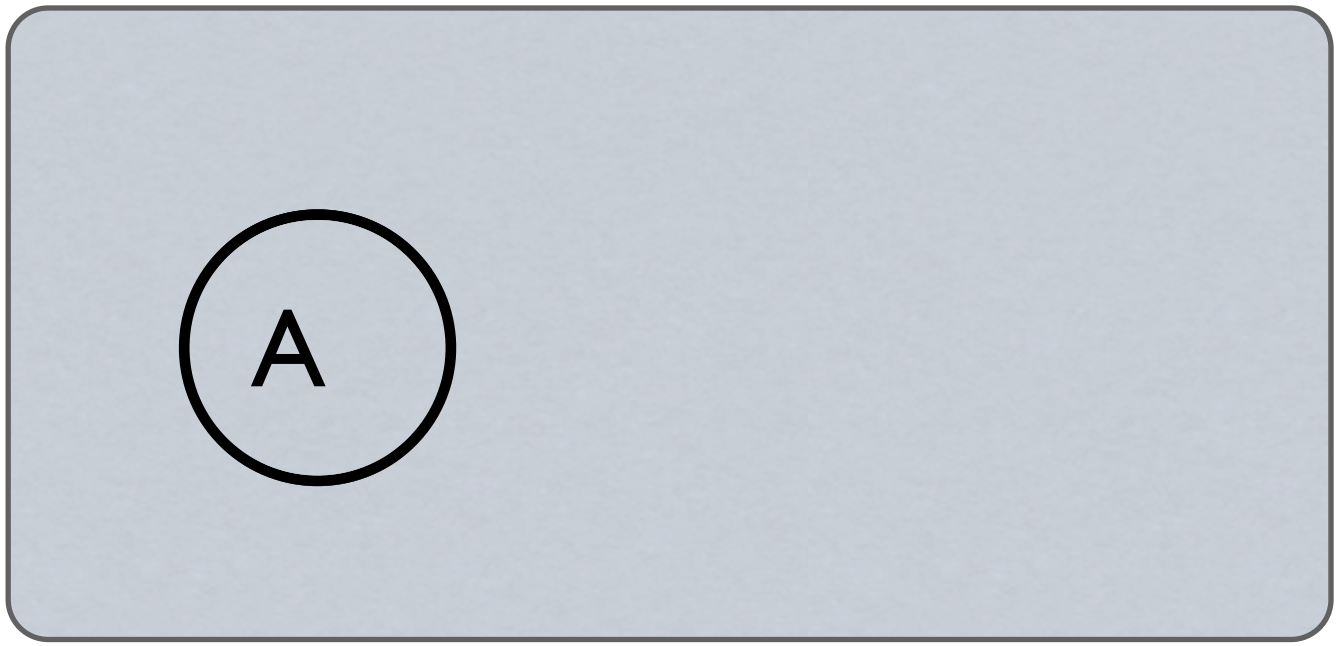
$$\langle \mathcal{O}(t)\rangle = \sum_{\alpha,\alpha'} c^*_{\alpha'} c_\alpha e^{i(E_{\alpha'}-E_\alpha)t} \langle \Psi_\alpha|\mathcal{O}|\Psi_\alpha\rangle \xrightarrow{t\to\infty} \sum_\alpha |c_\alpha|^2 \langle \Psi_\alpha|\mathcal{O}|\Psi_\alpha\rangle$$

If any eigenstate within the energy shell is thermal
[strong Eigenstate Thermalization Hypothesis (ETH)]
the generic state thermalizes!

No proof of strong ETH, but it is believed to hold for
"typical" non-integrable, translation-invariant systems…..
(some numerical evidences, but also some counter-examples)

Necessary Condition for ETH

Consider an infinite system



If the ETH is satisfied, the reduced density matrix must be thermal $\Rightarrow$

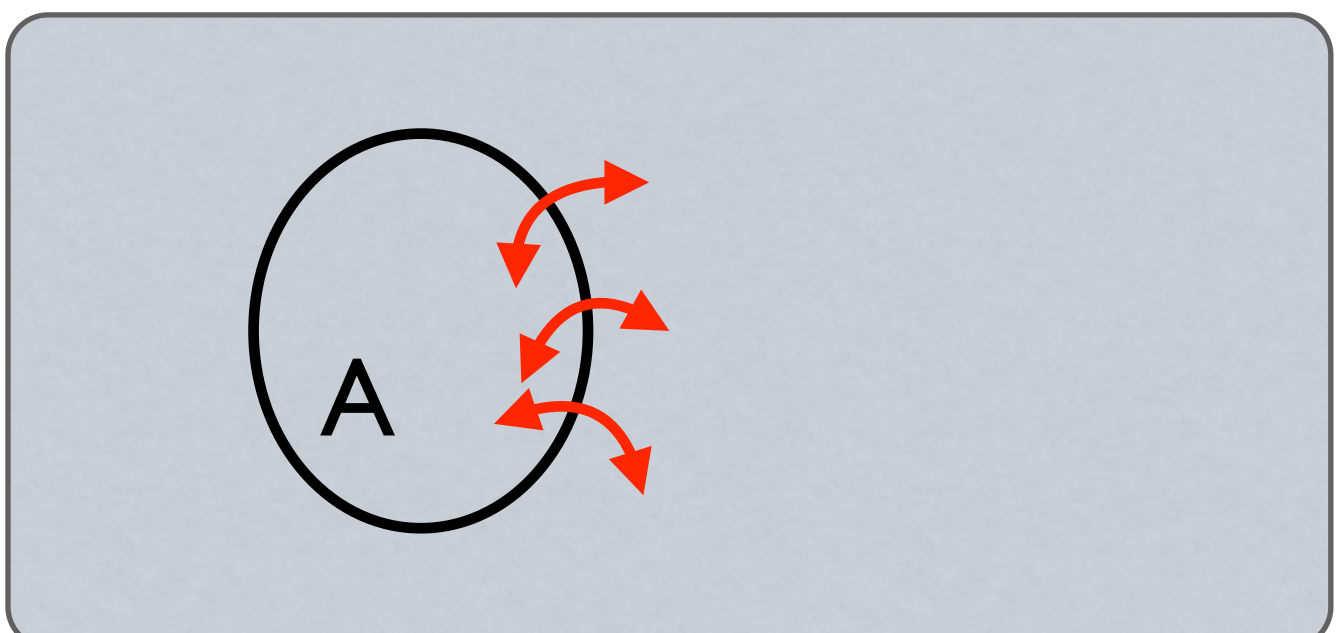
entanglement entropy must follow the volume law

Area Law

“Volume law” of the entanglement entropy implies the difficulty in numerical calculation

However, some states of physical interest has smaller entanglement

In particular, the entanglement entropy in the ground state of a typical “local” Hamiltonian is believed to follow the “area law”



$$S_E \propto \text{Area of } \partial A$$

(boundary length in 2D
constant in 1D)

Entanglement in 1D Ground States

$$|\Psi_0\rangle = \sum_{i_1, i_2, \ldots, i_n} c_{i_1 i_2 \ldots i_n} |i_1\rangle \otimes |i_2\rangle \ldots \otimes |i_n\rangle$$

Successive Schmidt decompositions

$$|\Psi\rangle = \sum_{\alpha=1}^{\chi_A} \lambda_\alpha |\Phi_\alpha^{[A]}\rangle \otimes |\Phi_\alpha^{[B]}\rangle$$

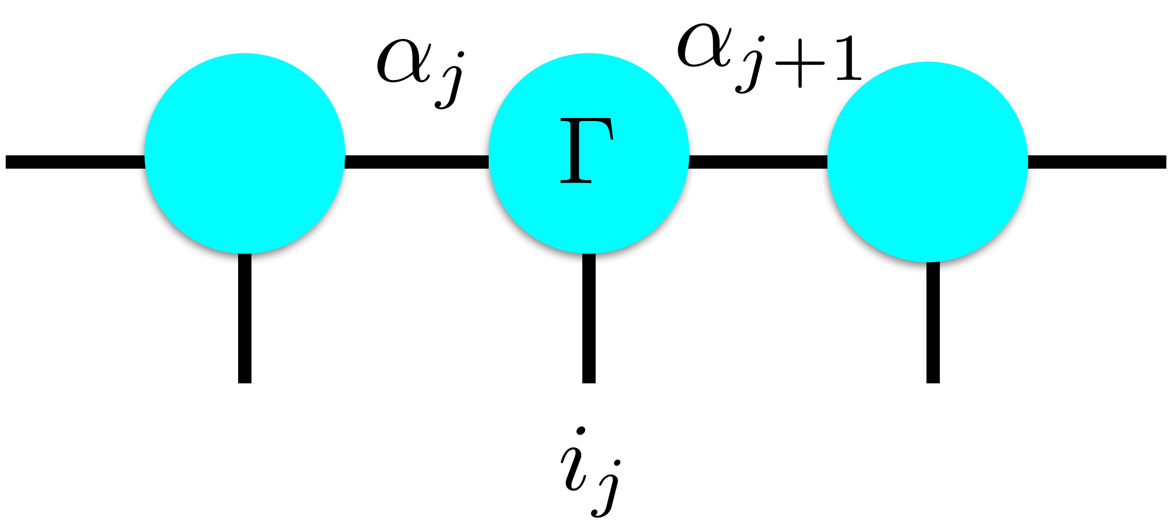
$$|\Psi_0\rangle = \sum_{\{\alpha_j\},\{i_j\}} \lambda_{\alpha_1} \Gamma^{i_1}_{\alpha_1 \alpha_2} \lambda_{\alpha_2} \Gamma^{i_2}_{\alpha_2 \alpha_3} \ldots |i_1\rangle \otimes |i_2\rangle \otimes \ldots$$

"Matrix Product State" AKLT 1987, Fannes-Nachtergale-Werner 1992, Vidal 2003, ….

Matrix Product States

$$|\Psi_0\rangle = \sum_{\{\alpha_j\},\{i_j\}} \lambda_{\alpha_1}\Gamma^{i_1}_{\alpha_1\alpha_2}\lambda_{\alpha_2}\Gamma^{i_2}_{\alpha_2\alpha_3}\ldots|i_1\rangle\otimes|i_2\rangle\otimes\ldots$$

"virtual (bond) indices"



$$\alpha_j = 1, 2, \ldots, \chi$$

$$m_j = 1, \ldots, d$$

"physical indices"

What does it mean?

Generic pure quantum state

$$|\Psi_0\rangle = \sum_{i_1,i_2,\ldots,i_n} c_{i_1 i_2 \ldots i_n} |i_1\rangle \otimes |i_2\rangle \ldots \otimes |i_n\rangle$$

number of parameters: d^L

d: dimension of local Hilbert space

(Translation-invariant) MPS

$$|\Psi_0\rangle = \sum_{\{\alpha_j\},\{i_j\}} \lambda_{\alpha_1} \Gamma^{i_1}_{\alpha_1\alpha_2} \lambda_{\alpha_2} \Gamma^{i_2}_{\alpha_2\alpha_3} \ldots |i_1\rangle \otimes |i_2\rangle \otimes \ldots$$

number of parameters: $d\chi^2 \ll d^L$!!

MPSs are very special among generic quantum states!

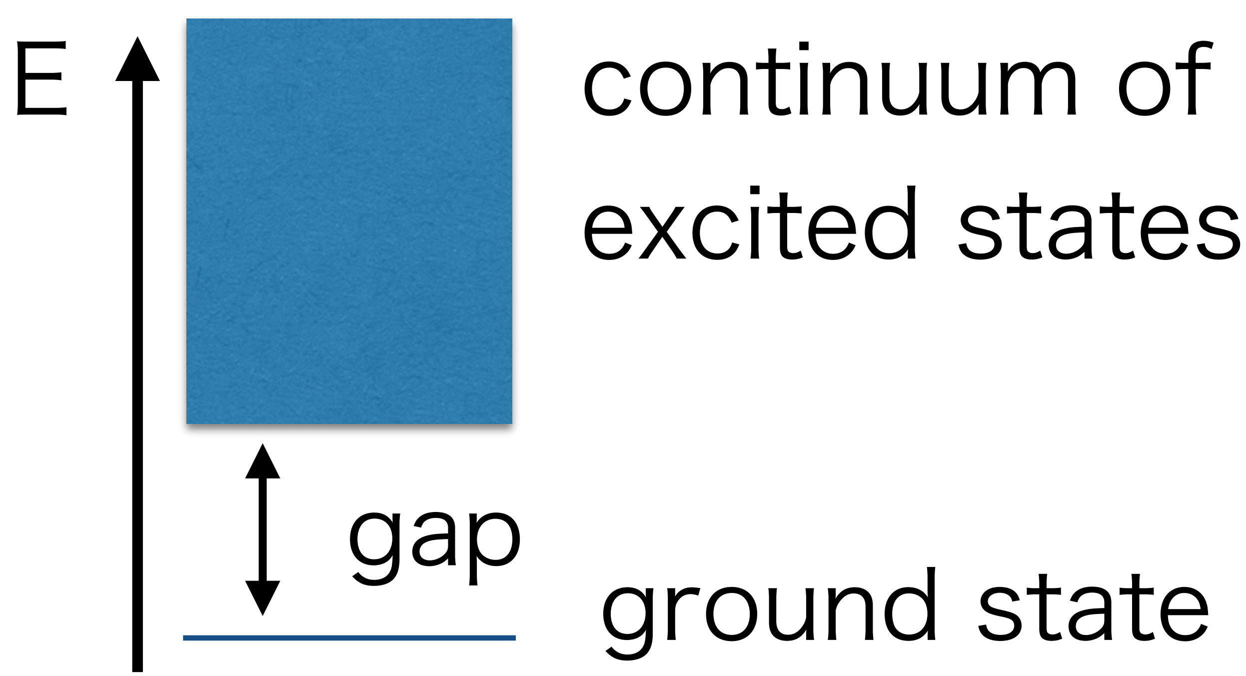
Significance of MPS

Significance of MPS:

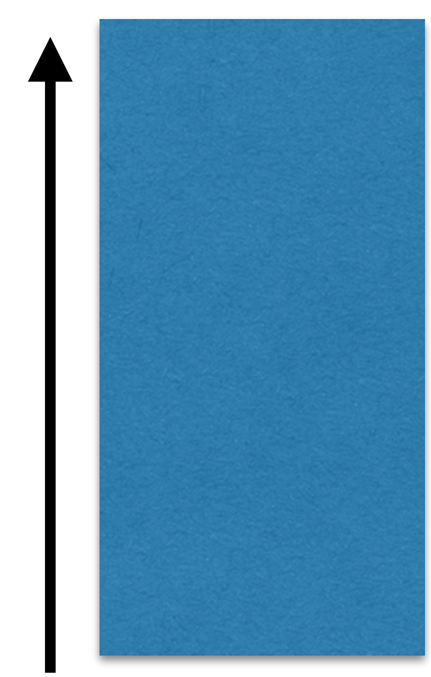
Any gapped ground state in 1D can be approximated by a MPS with a finite bond dimension χ

[Hastings 2007]

gapped (off-critical)

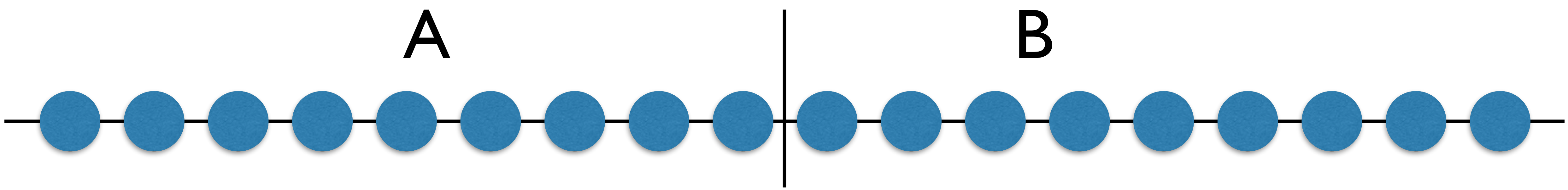


gapless (critical)



Why does MPS work??

Gapped ground states: finite correlation length



Bipartite entanglement entropy between A & B :

$S_E \sim \mathrm{constant}$ (“Area Law”)

even for an infinitely long system

Schmidt decomposition

$$|\Psi\rangle = \sum_\gamma \lambda_\gamma |\Psi^A_\gamma\rangle \otimes |\Psi^B_\gamma\rangle$$

need $O(e^{S_E}) \sim \mathrm{const.}$ terms in the sum!

Applications…

Density-Matrix Renormalization Group (DMRG)

powerful numerical approach to 1D quantum many-body problems

[White 1992]

DMRG = variational methods with MPS

The density-matrix renormalization group in the age of matrix product states

Ulrich Schollwöck

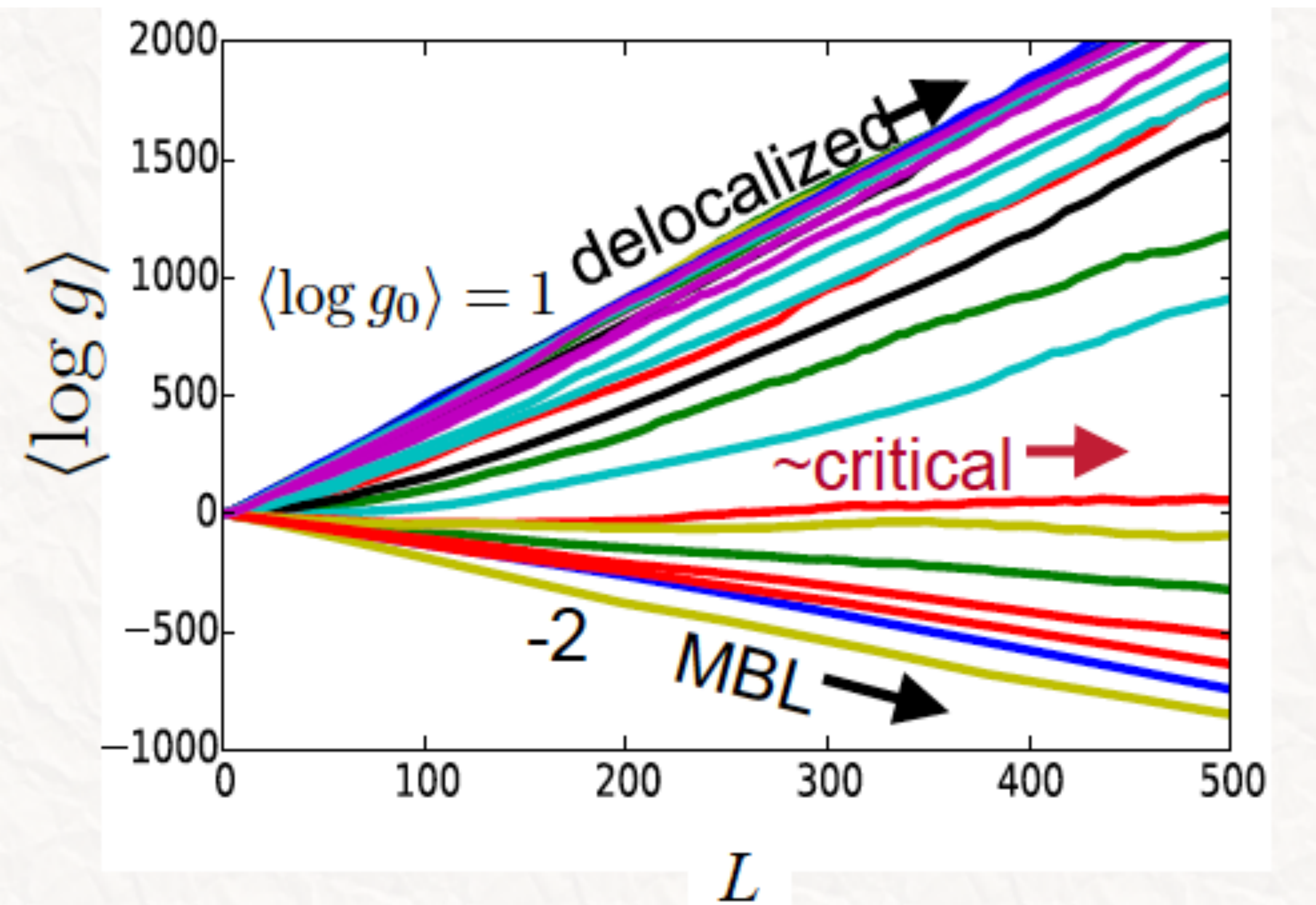
Arnold Sommerfeld Center for Theoretical Physics and Center for NanoScience, University of Munich, Theresienstrasse 37, 80333 Munich, Germany
Institute for Advanced Study Berlin, Wallotstrasse 19, 14159 Berlin, Germany

New perspectives, systematic improvements,…..

Many-Body Localization

Disordered interacting system ⇒ "Many-Body Localization"

Breakdown of thermalization $\sim$ breakdown of ETH



$\sim$eigenstate

entanglement

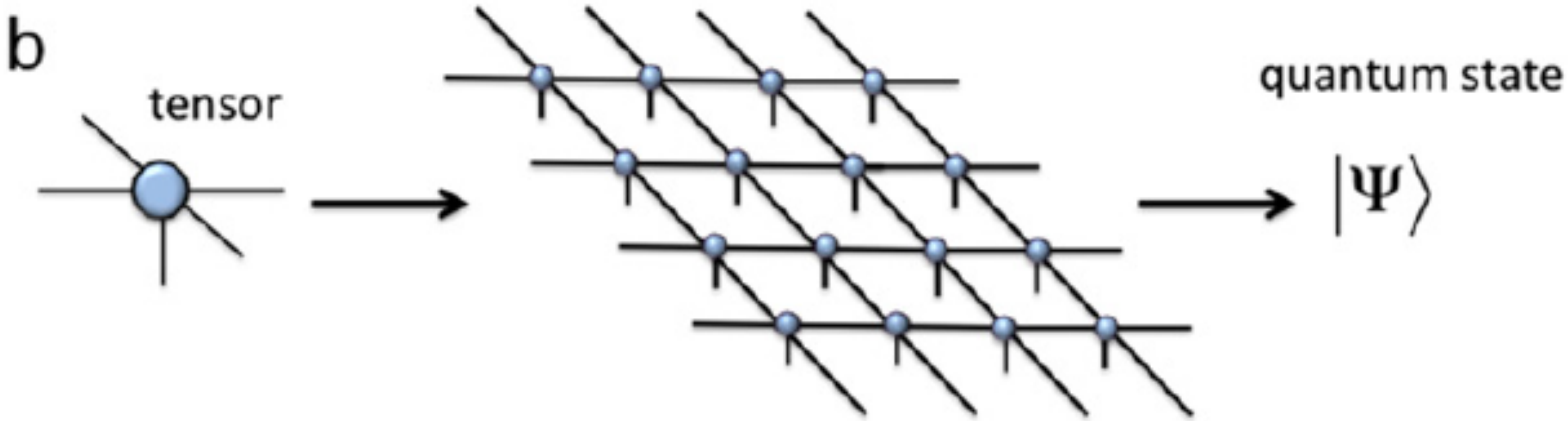
entropy

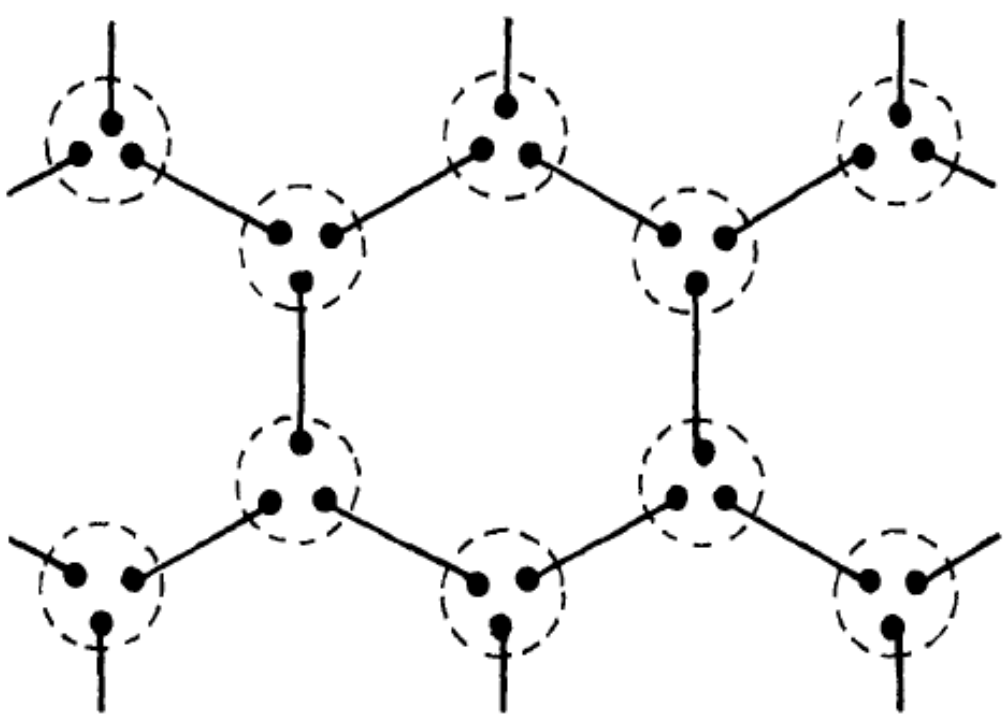
Vosk-Huse-Altman 2014

⇒ MPS also works in MBL phases even at finite energies!

Higher Dimensions

Generalization of MPS: "tensor network states"





Valence Bond Ground States in Isotropic Quantum Antiferromagnets

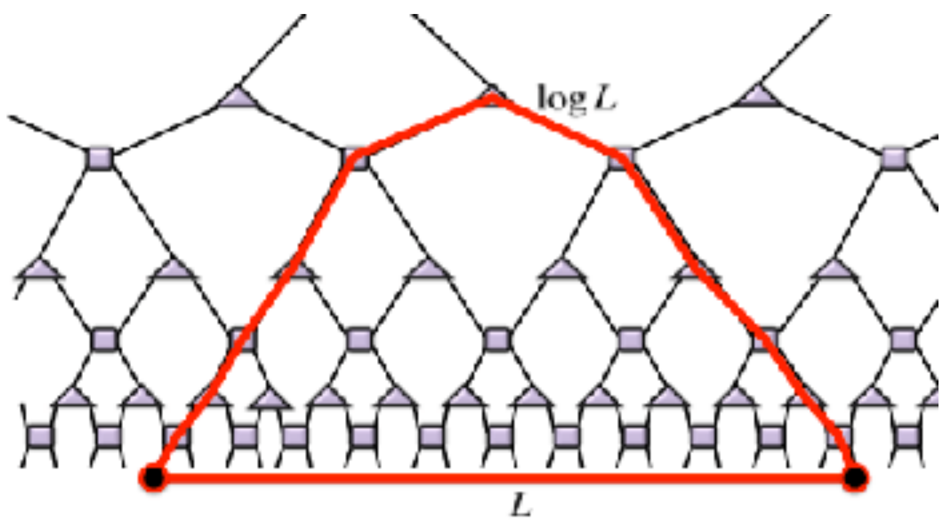
Ian Affleck[1,*], Tom Kennedy[2,**], Elliott H. Lieb[2,***], and Hal Tasaki[2,***]

[AKLT 1988]

Fig. 3.2. The VBS state on the hexagonal lattice. Each dot, line, and dotted circle represents a spin 1/2, a singlet pair, and the symmetrization of three spin 1/2's to create a spin 3/2

MERA

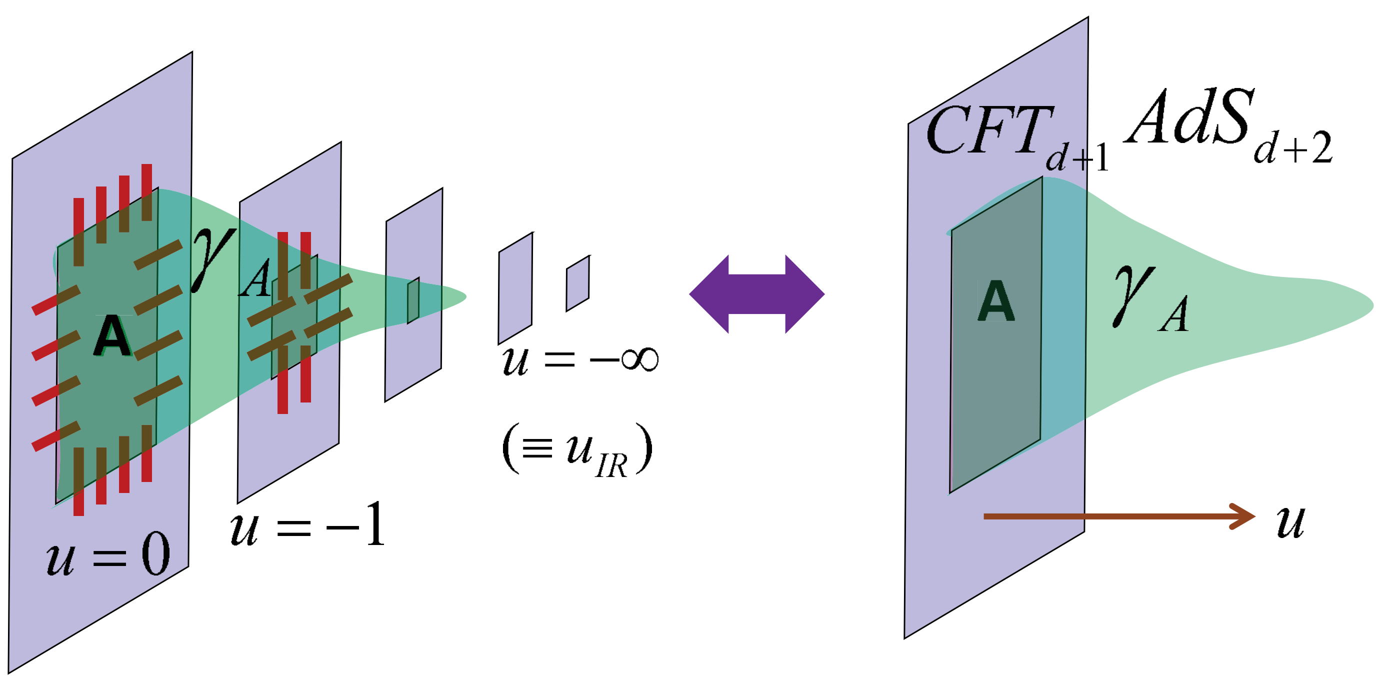
Multiscale Entanglement
Renormalization Ansatz
[Evenbly-Vidal 2009]



tree-like structure
for 1D critical states

$$S_E \propto \log L$$

MERA AdS/CFT

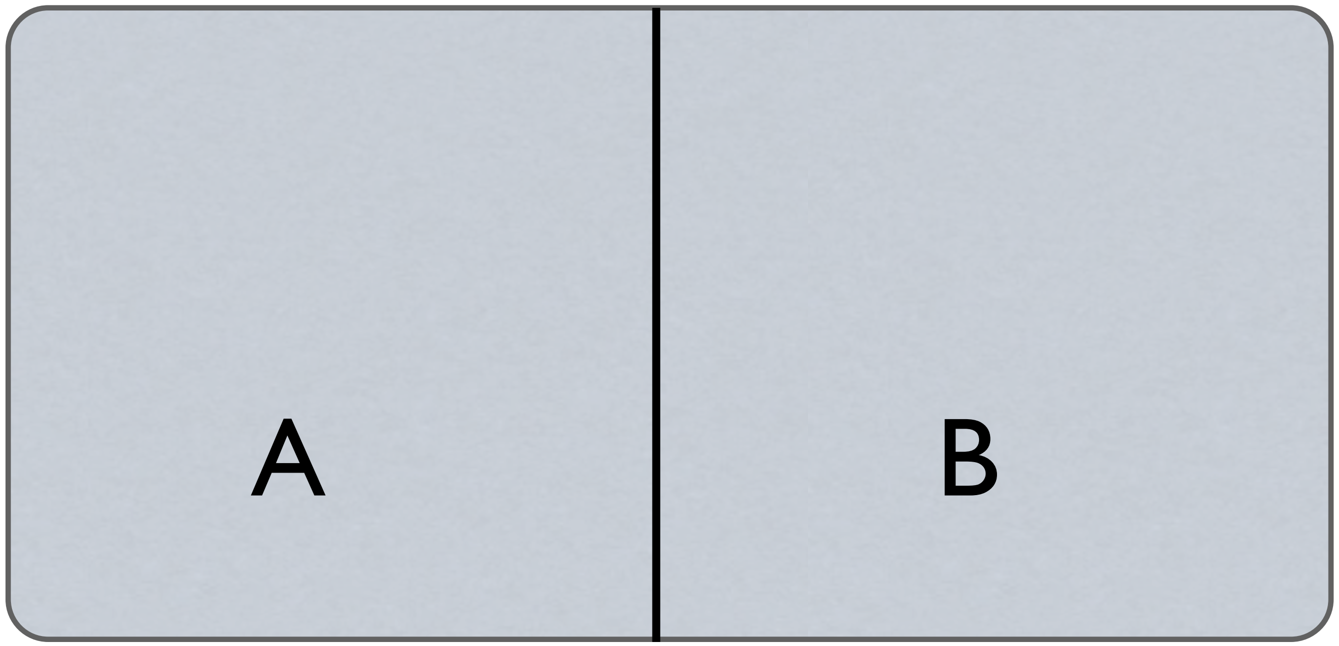


$S_A \propto \mathrm{Min}[\#\mathrm{Bonds}(\gamma_A)]$ $S_A \propto \mathrm{Min}[\mathrm{Area}]$

Relation to
gauge-gravity duality
(AdS/CFT)

Figure from
[Nozaki-Ryu-Takayanagi 2012]

Correction to Area Law



$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$

$$S_E = -\mathrm{Tr}\left[\rho_A \log \rho_A\right]$$

In 2D, the area law implies $S_E \sim \alpha \mathfrak{L}$

$\mathfrak{L}$: boundary length

α is non-universal constant, but there can be a universal correction of O(1)!

SSB phases

e.g. Ising model in the ordered phase

$$|\Psi_0\rangle \sim \frac{1}{\sqrt{2}}\left(|\uparrow\uparrow\ldots\uparrow\rangle + |\downarrow\downarrow\ldots\downarrow\rangle\right)$$

$$\Big\Downarrow$$

$$S_E \sim \alpha\mathfrak{L} + \log 2$$

"extra information"

Stephan-Furukawa-Misguich-Pasquier 2009

Ground-State Degeneracy

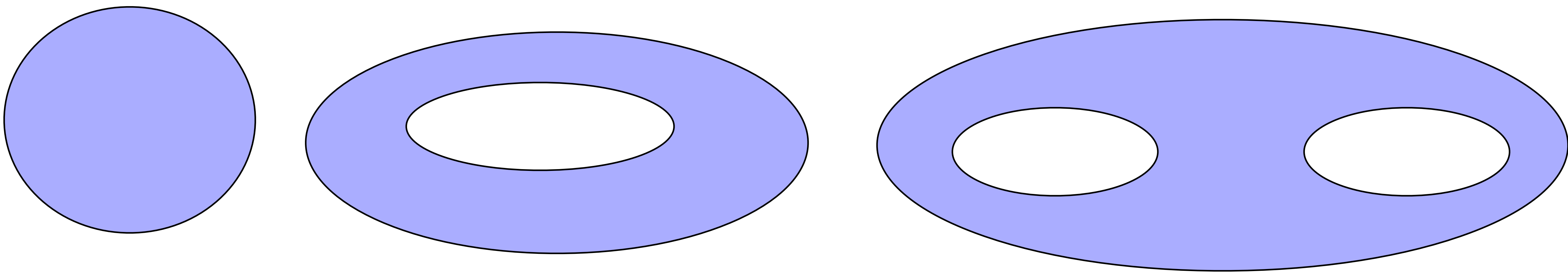
"Order" $\sim$ Spontaneous Symmetry Breaking

$$\Downarrow \quad \Uparrow ?$$

(non-accidental) ground-state degeneracy

Topological degeneracy (cf. FQH)

ground-state degeneracy N
depending on topology of the system



g=0 g=1 g=2

"genus" g: # of "holes" for the 2D surface

not a consequence of a ordinary SSB.....
a signature of a **topological order**!

degenerate g.s.: indistinguishable by any local operator

Frequently Asked Questions

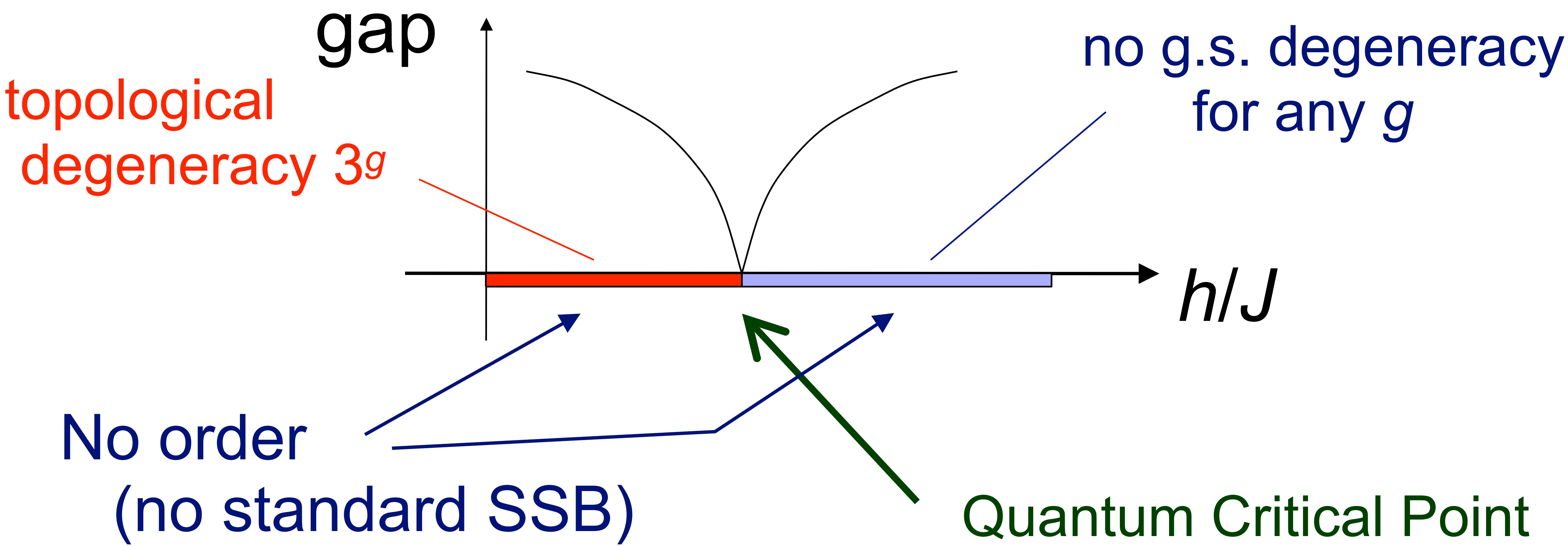
Q1: What is “topological order”?

Q2: Why should we care about the topological degeneracy?

I am not planning any kind of experiment with various genus *g* !

A: topological order itself cannot be “seen” directly, but is a useful concept behind several nontrivial physics (some of which may be measurable in experiments)

But sometimes…..



Why is there a QPT even though there is
no order in the both sides?
---- because one side has "topological order"

A "practical" definition

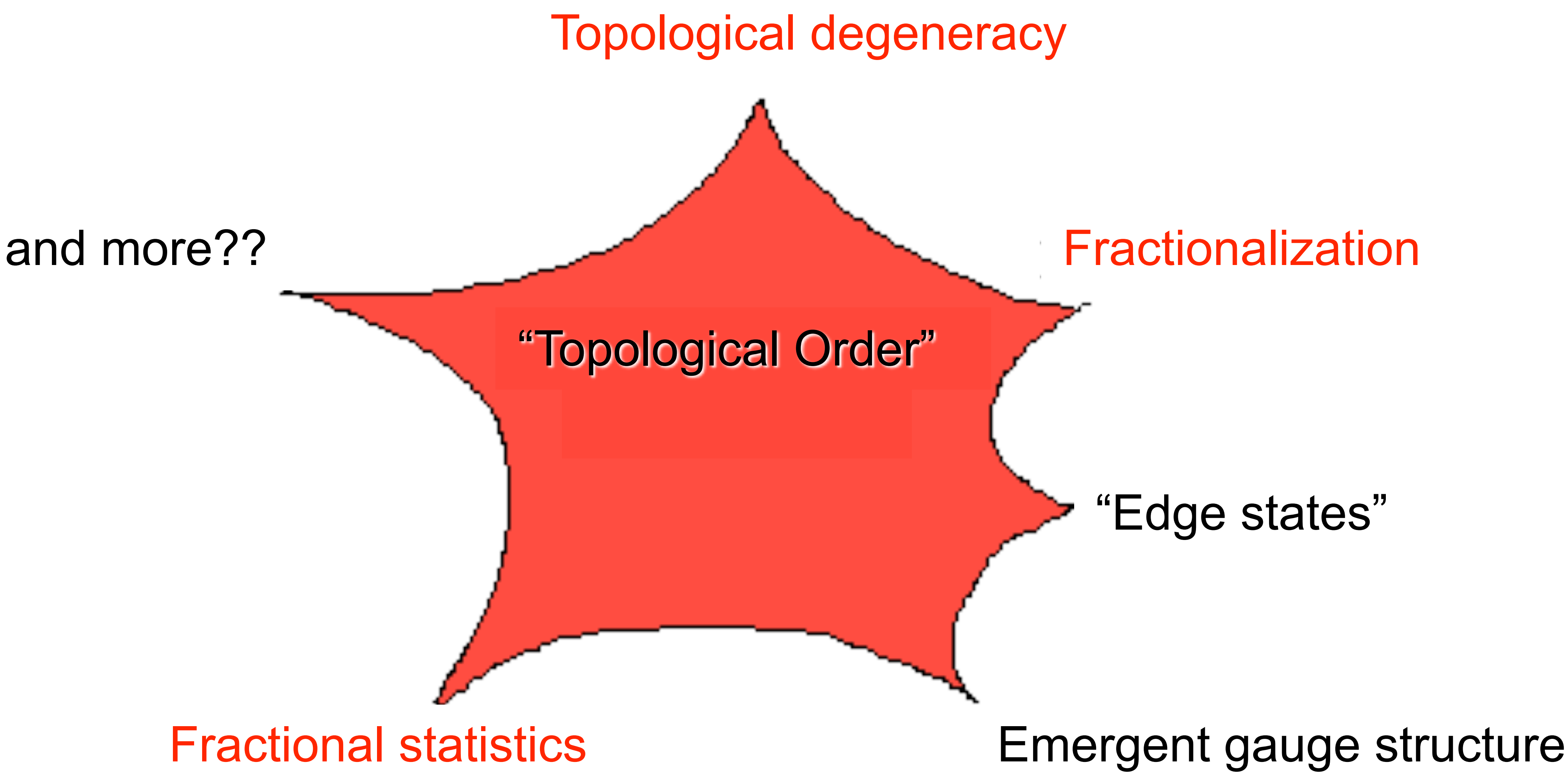
Separated by quantum phase transitions
= quantum phases

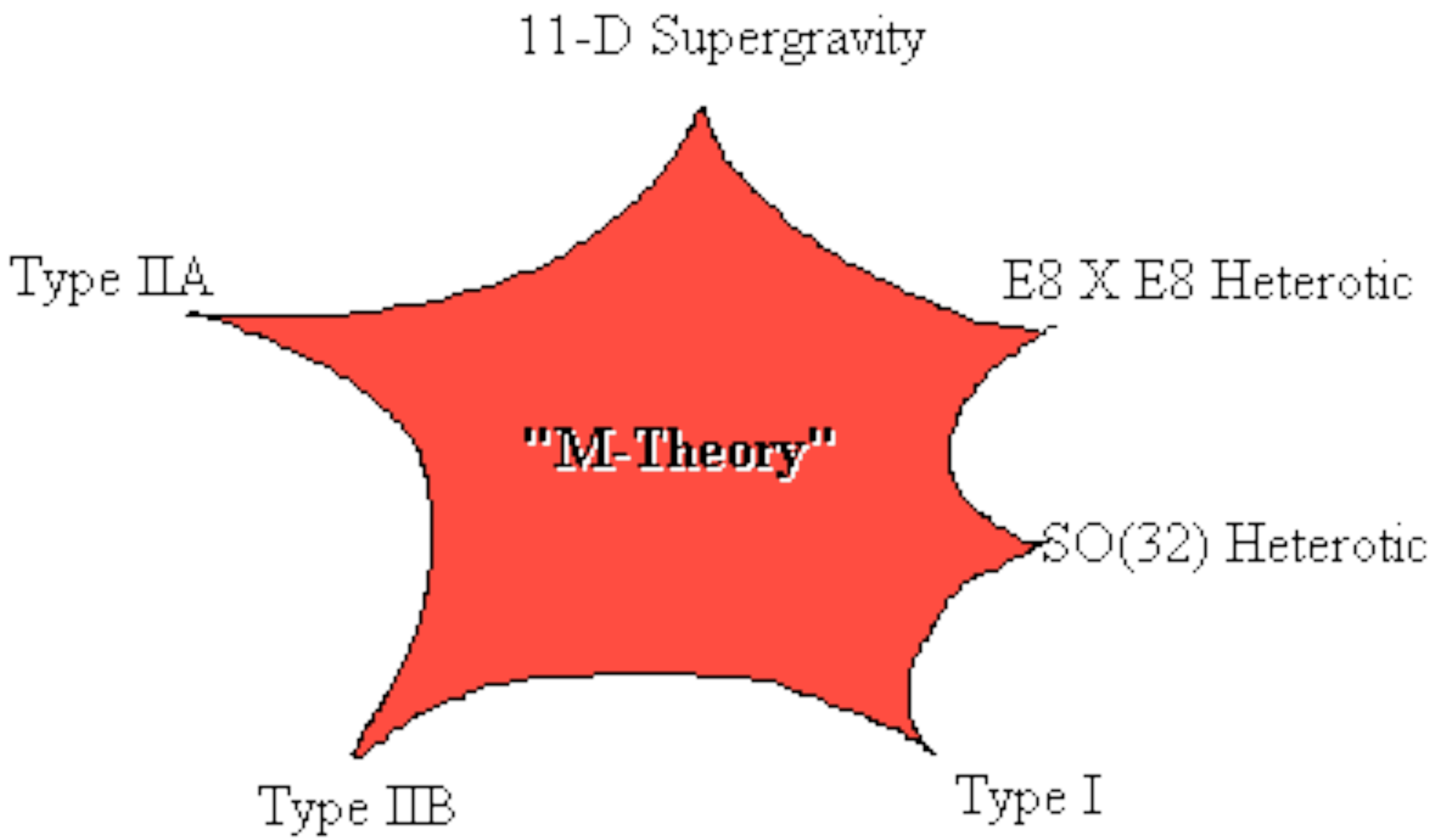
If a nontrivial phase cannot characterized by any local order parameter (or SSB), it has a "topological order"

Q: do we gain anything, compared to just knowing quantum phase transitions?

Q: what does "nontrivial phase" mean?

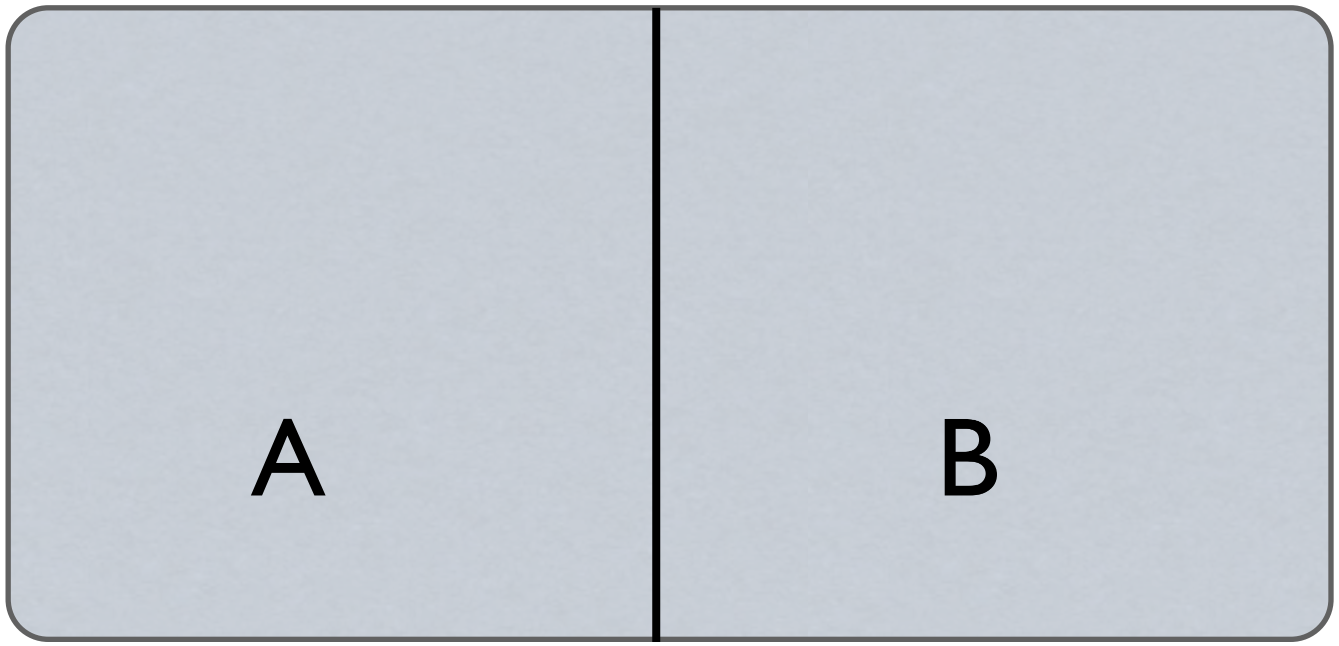
It appears that many ‘exotic’ physics are rather closely related.
“Topological order” is a (hypothetical) concept which is behind the exotica.





http://www.sukidog.com/jpierre/strings/mtheory.htm

Entanglement Entropy



$$\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$$

$$S_E = -\mathrm{Tr}\left[\rho_A \log \rho_A\right]$$

$$S_E \sim \alpha\mathfrak{L} - \gamma_{topo}$$

$\mathfrak{L}$: Boundary length

"area law" term
(non-universal
coefficient α)

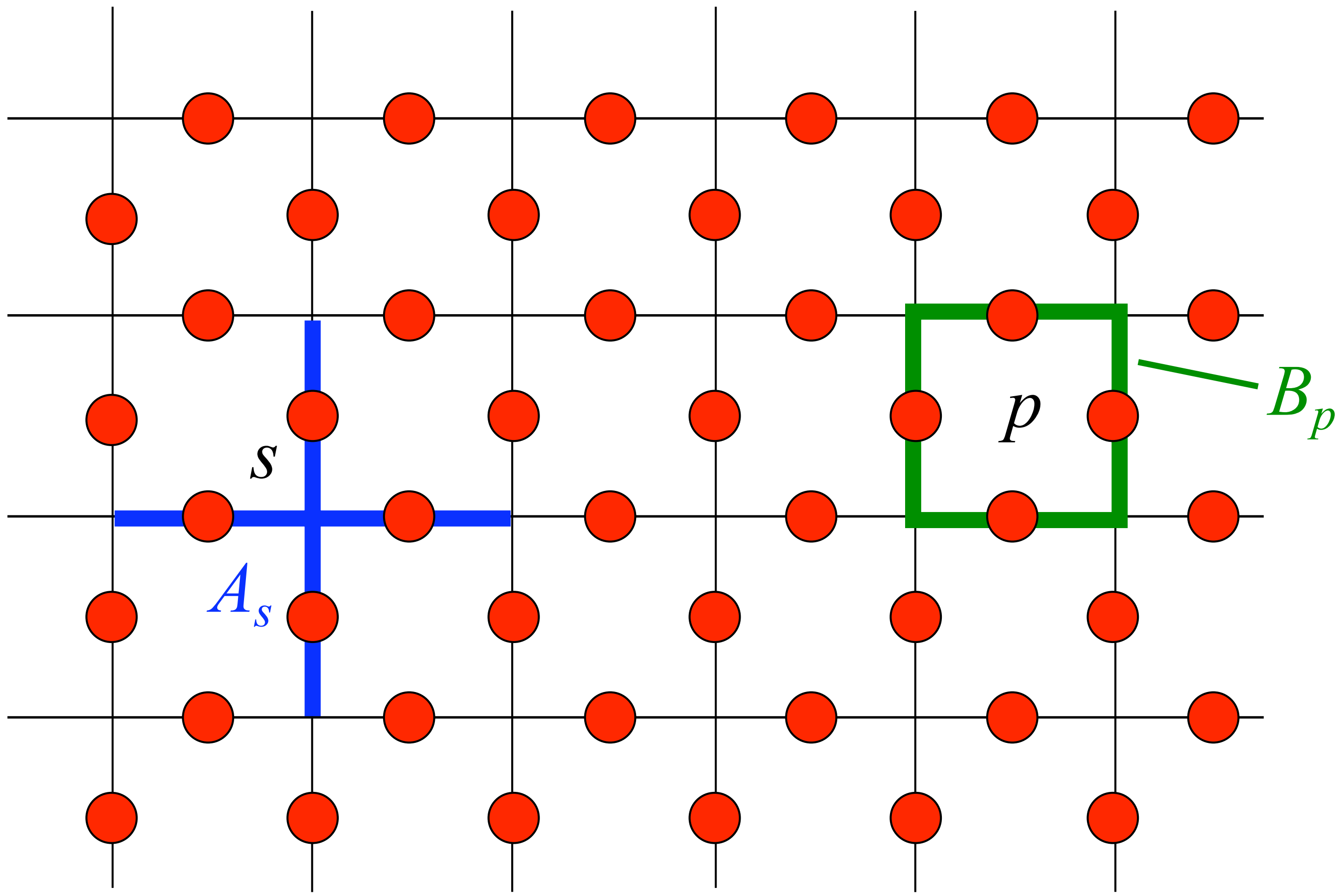
universal
"topological entanglement entropy"
Kitaev-Preskill / Levin-Wen 2006

encodes "topological order"
in the groundstate

Kitaev's toric code

Kitaev 1997

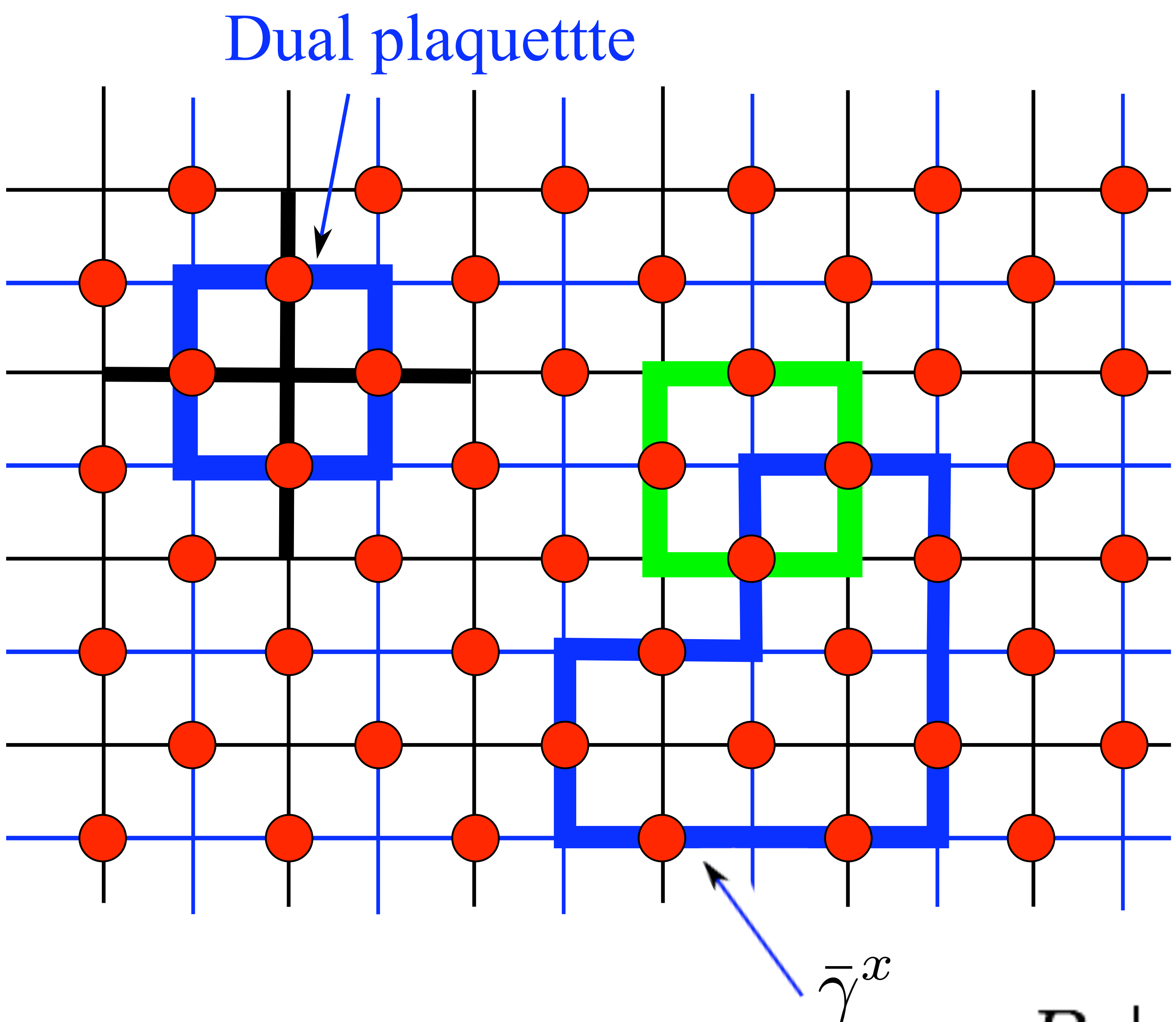
Exactly solvable model for "Z_2 topological phase"



$$\mathcal{H} = -\sum_s A_s - \sum_p B_p \qquad A_s = \prod_{j \in s} \sigma_j^x \qquad B_p = \prod_{j \in p} \sigma_j^z$$

$$(A_s)^2 = (B_p)^2 = 1 \qquad [A_s, B_p] = 0$$

Construction of GS



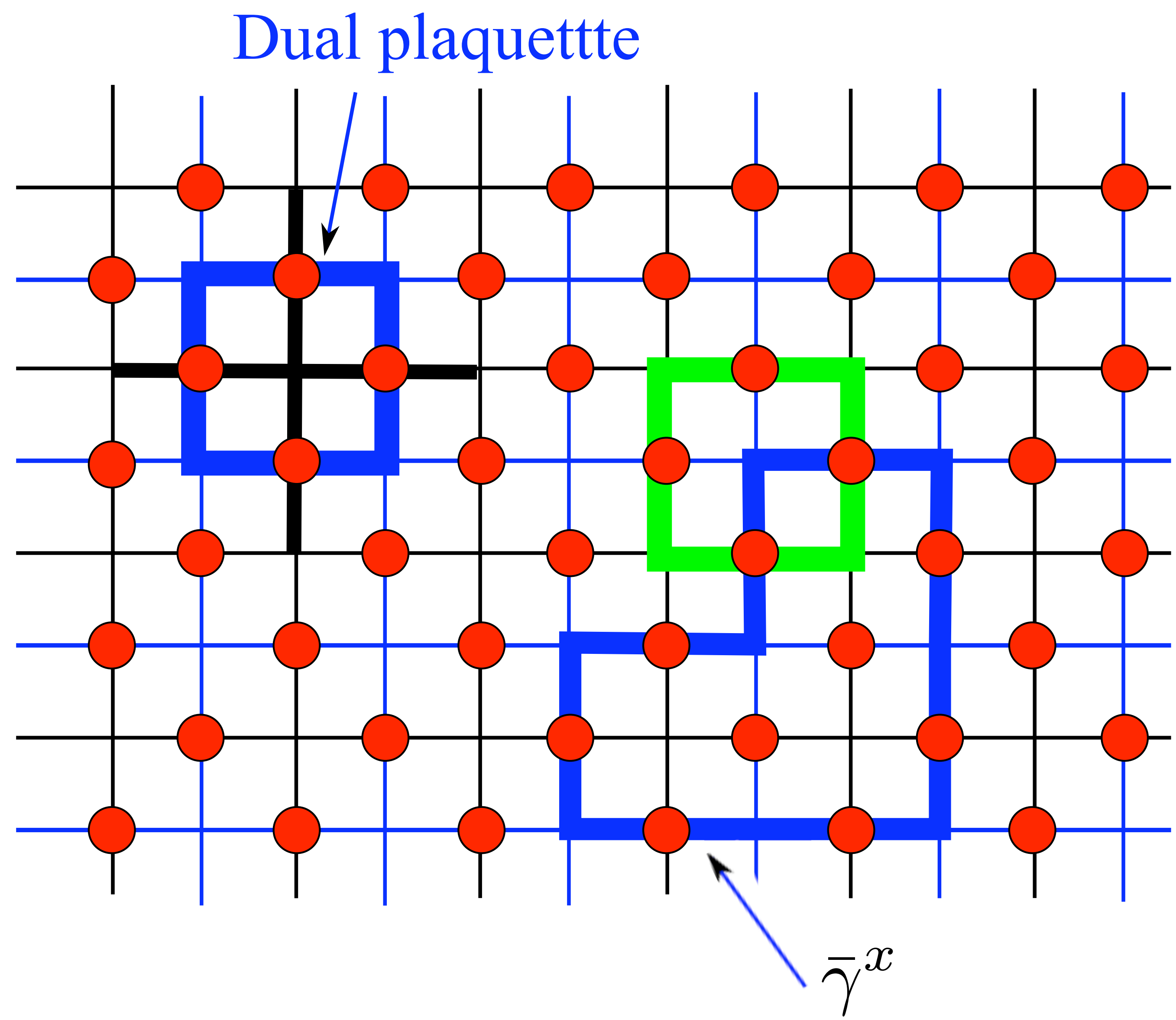
$$|\Psi\rangle \propto \sum_{\bar{\gamma}^x} W^x[\bar{\gamma}^x]|\mathrm{vac}_z\rangle$$

$$W^x[\bar{\gamma}^x] = \prod_{j\in\bar{\gamma}^x} \sigma_j^x$$

$$B_p|\mathrm{vac}_z\rangle = |\mathrm{vac}_z\rangle$$

$$[B_p, W^x[\bar{\gamma}^x]] = 0$$

$$B_p|\Psi\rangle = |\Psi\rangle$$

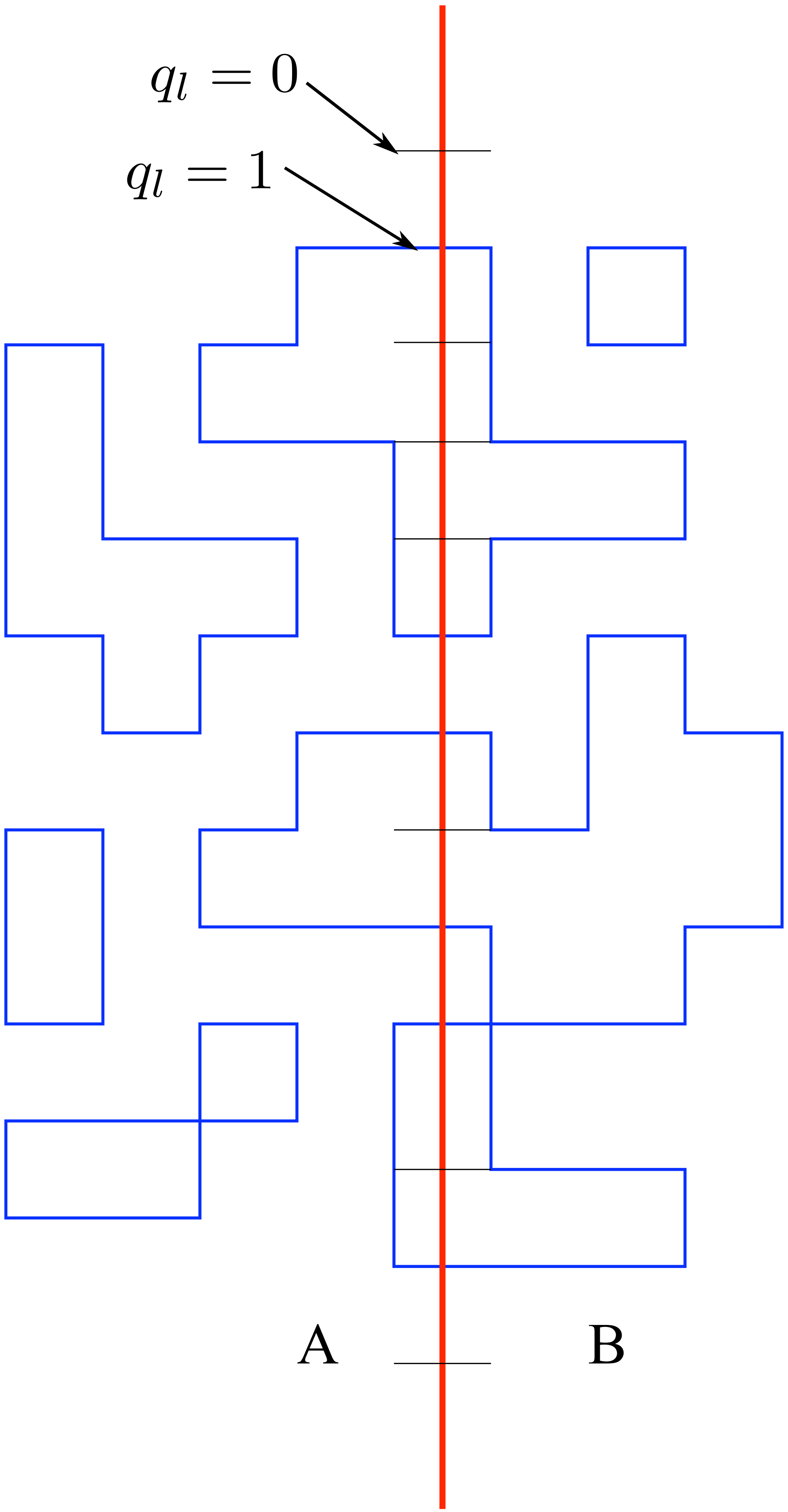


$$A_s W^x[\bar{\gamma}^x] = W^x[\bar{\gamma'}^x]$$

$$A_s|\Psi\rangle = A_s \sum_{\bar{\gamma}^x} W^x[\bar{\gamma}^x]|\mathrm{vac}_z\rangle = \sum_{\bar{\gamma'}^x} W^x[\bar{\gamma'}^x]|\mathrm{vac}_z\rangle = |\Psi\rangle$$

$|\Psi\rangle$ is a groundstate of $\mathcal{H} = -\sum_s A_s - \sum_p B_p$

Likewise, $|\Psi\rangle = \sum_{\bar{\gamma}^z} W^z[\bar{\gamma}^z]|\mathrm{vac}_x\rangle$ for $\bar{\gamma}^z$ closed-loops on original lattice



Bipartition of the system

Each “snapshot” of the groundstate
can be classified by
the string crossings
at the boundary Γ

Reduced Density Matrix

$$|\Psi\rangle = \frac{1}{\sqrt{N_q}} \sum_{\{q_l\}} |\Psi^A_{\{q_l\}}\rangle |\Psi^B_{\{q_l\}}\rangle$$

$|\Psi^R_{\{q_l\}}\rangle$ superposition of all the closed-loop strings states in region R (=A or B), for the given boundary configuration $\{q_l\}$

$$\rho^A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi| \propto \sum_{\{q_l\}} |\Psi^A_{\{q_l\}}\rangle\langle\Psi^A_{\{q_l\}}|.$$

Number of possible string configurations within region A:
= number of closed-loop string configurations completely contained in region A
independent of the boundary configuration $\{q_l\}$

(Hamma-Ioniciou-Zanardi 2005)

$S_E = \log N_q$ $\quad$ N_q: # of boundary configurations $\{q_l\}$

Entanglement Entropy

Each boundary link may be crossed (q_l=1) or not (q_l=0) by strings

$$N_q = 2^{\mathfrak{L}} \qquad ??$$

$\mathfrak{L}$: Boundary length (# of links on the boundary Γ)

In fact, the number of string crossings at the boundary must be even, since the strings form closed loops

$$N_q = 2^{\mathfrak{L}-1}$$

$$S_E = \log N_q = \mathfrak{L} \log 2 - \log 2$$

"area law" "topological EE"

Characterization of Topological Order

In general topologically ordered phase in 2+1 dimension:

$$S_E \sim \alpha \mathfrak{L} - \gamma_{topo}$$

$$\gamma_{topo} = \log D$$

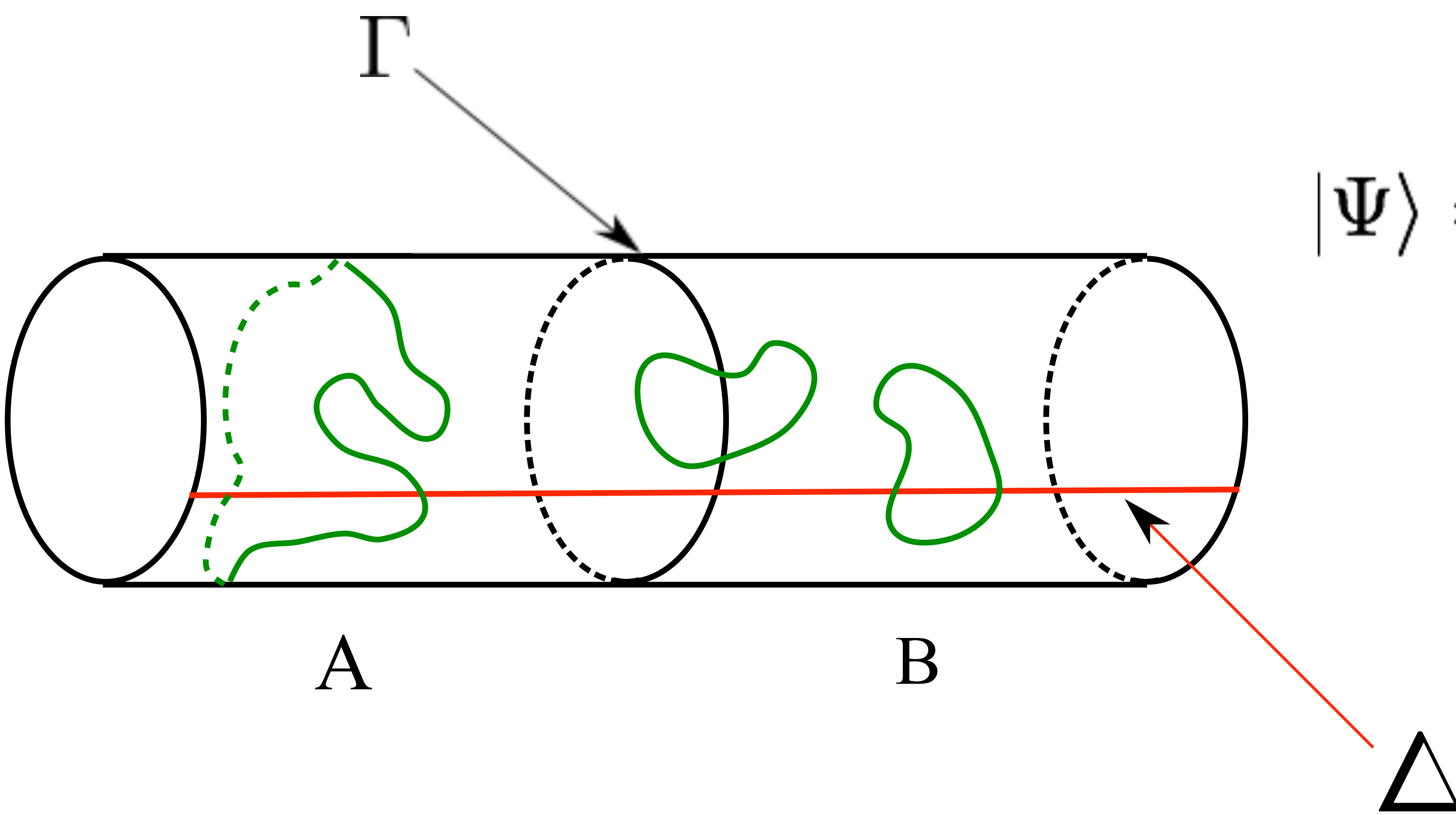
$$D = \sqrt{\sum_a {d_a}^2}$$ "total quantum dimension"

d_a: quantum dimension of anyon a

Topological Entanglement Entropy (TEE)

partially characterizes the topological order

Topological Degeneracy



$$|\Psi\rangle = \sum_{\bar{\gamma}^z} W^z[\bar{\gamma}^z] |\mathrm{vac}_x\rangle$$

Strings can form winding loops along the circumference

Winding number is conserved modulo 2

$\Rightarrow$ doubly degenerate GS **(topological degeneracy!)**

$|\xi_{0,1}\rangle$ winding number = 0,1 (modulo 2)

Topological Degeneracy & Qubit

Topologically degenerate groundstates could be used as a “qubit” which is robust against decoherence

The topologically degenerate groundstates are not connected by local operators $\Rightarrow$ suppression of decoherence

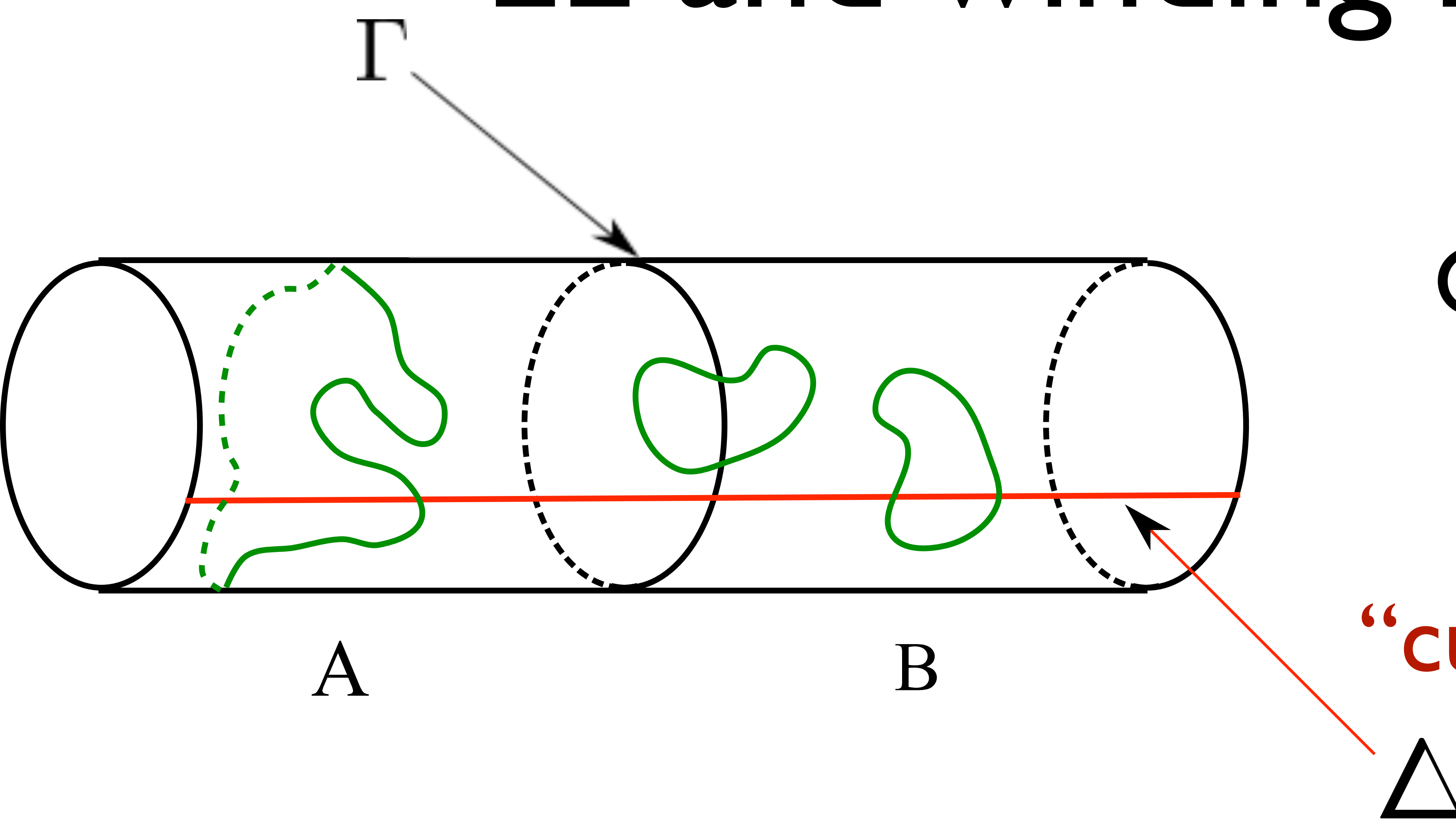
(manipulation & measurement become nontrivial, too)

Error correction procedure

$\Rightarrow$ **“toric code”**

Kitaev 1997

EE and winding number



Consider one of the "pure" GS $|\xi_{0,1}\rangle$

Crossing # at the cut = winding # (mod 2)

$$|\xi_1\rangle = \frac{1}{\sqrt{2N_q}} \sum_{\{q_l\}} \left(|\Psi^A_{\{q_l\},0}\rangle |\Psi^B_{\{q_l\},1}\rangle + |\Psi^A_{\{q_l\},1}\rangle |\Psi^B_{\{q_l\},0}\rangle \right)$$

crossing # at the cut (mod 2) within A or B

$$S_E = \mathfrak{L} \log 2$$

TEE vanishes, owing to the extra entanglement

EE in general groundstate

$$|\Psi\rangle = c_0|\xi_0\rangle + c_1|\xi_1\rangle$$

$$S_E = \mathfrak{L}\log 2 - [\log 2 - S_{cl}(\{\tilde{p}_0, \tilde{p}_1\})]$$

$$\tilde{p}_0 = \frac{|c_0 + c_1|^2}{2}$$

$$\tilde{p}_1 = \frac{|c_0 - c_1|^2}{2}$$

TEE dependence on the GS

pointed out for special cases by Hamma-Ioniciou-Zanardi 2005

$$S_{cl}(\{p_\mu\}) = -\sum_\mu p_\mu \log p_\mu$$

"classical entropy"

$$0 \le S_{cl}(\{\tilde{p}_0, \tilde{p}_1\}) \le \log 2$$

$$0 \le \gamma^{(\mathrm{cylinder})} \le \log 2$$

Dong-Fradkin-Leigh-Nowling 2008

Zhang-Grover-Turner-MO-Vishwanath 2012

EE on torus

Similar analysis of EE on torus (4 topologically degenerate g.s.):

TEE dependence on the choice of the ground state

↓

"modular S-matrix"

↓

Mutual statistics of anyons

More detailed characterization of the topological order!

Summary

(Quantum) information gives a useful perspective in old and new problems in condensed matter physics

some examples:

formulation of tensor-network states

improvements of tensor-network based numerical methods

characterization of topological states in terms of entanglement

This can also help developments in condensed matter (such as topologically protected qubits) also useful for quantum information processing