Smoothed α-Renyi divergences and divergence rates

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NII

Stability of KL-divergence

$$D(\rho\|\sigma) := \mathrm{tr}\,\rho(\ln\rho - \ln\sigma)$$ **KL-divergence**

$$\lim_{\varepsilon\downarrow 0}\ \overline{\lim_{n\to\infty}}\ \inf_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} D(\rho'\|\sigma^{\otimes n}) = D(\rho\|\sigma)$$

downward smoothing

$$\lim_{\varepsilon\downarrow 0}\ \underline{\lim_{n\to\infty}}\ \sup_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} D(\rho'\|\sigma^{\otimes n}) = D(\rho\|\sigma)$$

upward smoothing

The order of the limit matters!
Same as shrinking ε very very slowly in n

How this generalizes to similar quantities ?

Renyi divergence

α-Divergence

Classical version

$$D_\alpha(p||q) := \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sum_x (p(x))^\alpha \big(q(x)\big)^{1-\alpha}$$

Relation to KL-divergence

$$\lim_{\alpha \to 1} D_\alpha(p||q) = D(q||p)$$

Importance in infomration theory

Evaluation of error exponents of state discremination

α-Divergence

Classical version

$$D_\alpha(p||q) := \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sum_x (p(x))^\alpha \big(q(x)\big)^{1-\alpha}$$

Quantum version (1)

$$D_{\alpha,mes}(\rho||\sigma) := \max_M D_\alpha(p^M||q^M)$$

$$= \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sigma \left(\frac{T_0}{\alpha}\right)^\alpha$$

[M10]

$$\mathrm{D} f_\alpha^\#(T_0)[\sigma] = \rho$$

D: Frechet derivative

$$f_\alpha^\#(t) := \begin{cases} \mathrm{sgn}(\alpha)\cdot(1-\alpha)|\alpha|^{\frac{-\alpha}{\alpha-1}} t^{\frac{\alpha}{\alpha-1}}, & t \geq 0, \\ \infty, & t \leq 0. \end{cases}$$

α-Divergence

Classical version

$$D_\alpha(p||q) := \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sum_x (p(x))^\alpha \big(q(x)\big)^{1-\alpha}$$

Quantum version (2)

$$D_{\alpha,\mathrm{asy}}(\rho||\sigma) := \lim \frac{1}{n} D_{\alpha,\mathrm{asy}}(\rho^{\otimes n}||\sigma^{\otimes n})$$

$$= \begin{cases} \frac{\mathrm{sgn}(\alpha)}{\alpha-1} \ln \ \mathrm{tr}\ \left(\sigma^{(1-\alpha)/2\alpha}\rho\sigma^{(1-\alpha)/2\alpha}\right)^\alpha, & (\alpha \in [\frac{1}{2},\infty)) \\ \frac{\mathrm{sgn}(\alpha)}{\alpha-1} \ln \ \mathrm{tr}\ \left(\rho^{\frac{\alpha}{2(1-\alpha)}}\sigma\rho^{\frac{\alpha}{2(1-\alpha)}}\right)^{1-\alpha}, & (\alpha \in (-\infty,\frac{1}{2}]) \end{cases}$$

[OgawaMosonyi 10]

α-Divergence

Classical version

$$D_\alpha(p||q) := \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sum_x (p(x))^\alpha \big(q(x)\big)^{1-\alpha}$$

Quantum version (3)

$$D_{\alpha,gen}(\rho||\sigma) := \min\{D^\alpha(p\|q); \Gamma(p) = \rho, \Gamma(q) = \sigma, \Gamma\text{:CPTP}\}$$

$$= \frac{\mathrm{sgn}(\alpha)}{\alpha - 1} \ln \mathrm{tr}\, \sigma \left(\sigma^{-1/2} \rho \sigma^{-1/2}\right)^\alpha, \ (\alpha \in (-1, 2) \cup \{\infty\})$$

[M10]

α-Divergence

Classical version

$$D_\alpha(p||q) := \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sum_x (p(x))^\alpha \left(q(x)\right)^{1-\alpha}$$

Quantum version (4)

$$D_{\alpha,P}\,(\rho||\sigma) := \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \mathrm{tr}\, \rho^\alpha \sigma^{1-\alpha} \quad (-1 \le \alpha \le 2)$$

[Petz ?]

Appears in Hoeffeding bound
[Hayashi][Nagaoka]

α-Divergence

Classical version

$$D_\alpha(p||q) := \frac{\mathrm{sgn}(\alpha)}{1-\alpha} \ln \sum_x (p(x))^\alpha \big(q(x)\big)^{1-\alpha}$$

Quantum versions

(Consistent)

Coincide with classical version if states commutes

(CPTP Monotone)

$$D_{\alpha,Q}\,(\rho||\sigma) \geq D_{\alpha,Q}\,(\Lambda(\rho)||\Lambda(\sigma))$$

Properties of Quantum Renyi

$$D_{\alpha,mes}(\rho||\sigma) \leq D_{\alpha,Q}(\rho||\sigma) \leq D_{\alpha,gen}(\rho||\sigma)$$

$$D_{\alpha,mes} \leq D_{\alpha,P} \leq D_{\alpha,asy} \leq D_{\alpha,gen}$$

$$D_{0,mes} = D_{0,P} = D_{0,asy} \lneqq D_{0,gen}$$

$$D_{\infty,mes} = D_{\infty,P} = D_{\infty,asy} = D_{\infty,gen}$$

$D_{0,mes}$ coicside with min-relative entropy of Renner-Datta

$D_{\infty,gen}$ max-relative entropy

Smoothing α-Renyi **α=0,∞** (i.i.d)

$$\lim_{\varepsilon\downarrow 0}\ \overline{\lim_{n\to\infty}}\ \inf_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0}\ \frac{1}{n}D_{\infty,gen}\left(\rho'\middle\|\sigma^{\otimes n}\right)=D(\rho\|\sigma)$$

Downward smoothing

$$\lim_{\varepsilon\downarrow 0}\ \underline{\lim_{n\to\infty}}\ \sup_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0}\ \frac{1}{n}D_{0,mes}\left(\rho'\middle\|\sigma^{\otimes n}\right)=D(\rho\|\sigma)$$

upward smoothing

DattaRenner 05

Downward-smoothing (i.i.d)

KL divergence is stable

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{n \to \infty}} \inf_{\rho' \in B^1_\varepsilon(\rho^n), \rho' \geq 0} D(\rho' \| \sigma^{\otimes n}) = D(\rho \| \sigma)$$

Renyi divergence is not stable

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{n \to \infty}} \inf_{\rho' \in B^1_\varepsilon(\rho^n), \rho' \geq 0} D_{\alpha,Q}(\rho' \| \sigma^{\otimes n}) = D(\rho \| \sigma) \quad (\alpha > 1)$$

$$= 0 \quad (\alpha < 1)$$

Does not depends on choice of quantum analogue

Upward-smoothing (i.i.d)

KL divergence is stable

$$\lim_{\varepsilon\downarrow 0}\ \underline{\lim}_{n\to\infty}\ \sup_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} D(\rho'\|\sigma^{\otimes n}) = D(\rho\|\sigma)$$

Renyi divergence

$$\lim_{\varepsilon\downarrow 0}\ \underline{\lim}_{n\to\infty}\ \sup_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} D_{\alpha,Q}(\rho'\|\sigma^{\otimes n}) = -\ln s_{min}(\sigma) \quad (\alpha > 1)$$

$$= \infty \quad (\alpha < 0)$$

Does not depends on choice of quantum analogue

How about $0 < \alpha < 1$?

upward-smoothing, $\mathbf{0 < \alpha < 1}$ (i.i.d)

Recall : $D_{\alpha,mes} \leq D_{\alpha,P} \leq D_{\alpha,asy} \leq D_{\alpha,gen}$

$$\lim_{\varepsilon \downarrow 0} \varliminf_{n \to \infty} \sup_{\rho' \in B_\varepsilon^1(\rho^n), \rho' \geq 0} D_{\alpha,mes}(\rho' \| \sigma^{\otimes n}) = D(\rho \| \sigma)$$

If $\alpha \leq \frac{1}{2}$, any $D_{\alpha,Q} \leq D_{\alpha,asy}$ satisfies above.

$$\because D_{\alpha,asy} \leq D_{\frac{1}{2},asym} = D_{\frac{1}{2},mes}$$

some dependency on
choice of quantum analogue ?

Beyond i.i.d: Divergence rate

Quantum information spectrum by Nagaoka is defined using only order structure.

Can easily generalize to GPT.

Their meanings in Hypothesis test also generalizes

DEF

$$\overline{D}(\{\rho^n\}\|\{\sigma^n\}) := \inf\{\lambda; \lim \mathrm{tr}\left(\rho^n - 2^{n\lambda}\sigma^n\right)_+ = 0\}$$

$$\underline{D}(\{\rho^n\}\|\{\sigma^n\}) := \inf\{\lambda; \lim \mathrm{tr}\left(\rho^n - 2^{n\lambda}\sigma^n\right)_+ = 1\}$$

$(A)_+$: positive part

$$\overline{D}(\{\rho^{\otimes n}\}\|\{\sigma^{\otimes n}\}) = \underline{D}(\{\rho^{\otimes n}\}\|\{\sigma^{\otimes n}\}) = D(\rho\|\sigma)$$

$$D(\rho\|\sigma) = \mathrm{tr}\,\rho(\ln\rho - \ln\sigma)$$

Smoothing α-Renyi α=0,∞

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{n\to\infty}} \inf_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} \frac{1}{n} D_{\infty,gen}(\rho'\|\sigma^n) = \overline{D}(\{\rho^n\}\|\{\sigma^n\})$$

$$\lim_{\varepsilon\downarrow 0} \underline{\lim_{n\to\infty}} \sup_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} \frac{1}{n} D_{0,mes}(\rho'\|\sigma^n) = \underline{D}(\{\rho^n\}\|\{\sigma^n\})$$

DattaRenner 05

Downward-smoothing (general)

Renyi divergence is not stable

$$\lim_{\varepsilon \downarrow 0} \overline{\lim_{n \to \infty}} \inf_{\rho' \in B_\varepsilon^1(\rho^n), \rho' \geq 0} D_{\alpha,Q}(\rho' \| \sigma^n) = \overline{D}(\{\rho^n\} \| \{\sigma^n\}) \qquad (\alpha > 1)$$

$$= 0 \qquad (\alpha < 1)$$

Does not depends on choice of quantum analogue

Upward-smoothing (general)

Renyi divergence

$$\lim_{\varepsilon\downarrow 0} \underline{\lim}_{n\to\infty} \sup_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} D_{\alpha,Q}(\rho'\|\sigma^n) \quad = \underline{\lim}_{n\to\infty} \frac{-1}{n} \ln\, s_{\min}(\sigma^n)$$

$$(\alpha > 1)$$

$$= \infty \qquad (\alpha < 0)$$

Does not depends on choice of quantum analogue

How about $0 < \alpha < 1$?

upward-smoothing, $\mathbf{0} < \boldsymbol{\alpha} < \mathbf{1}$ (general)

Recall : $D_{\alpha,mes} \leq D_{\alpha,P} \leq D_{\alpha,asy} \leq D_{\alpha,gen}$

$$\lim_{\varepsilon \downarrow 0} \underline{\lim}_{n \to \infty} \sup_{\rho' \in B^1_\varepsilon(\rho^n), \rho' \geq 0} D_{\alpha,mes}(\rho' \| \sigma^n) = \underline{D}(\{\rho^n\} \| \{\sigma^n\})$$

If $\alpha \leq \frac{1}{2}$, any $D_{\alpha,Q} \leq D_{\alpha,asy}$ satisfies above.

some dependency on
choice of quantum analogue ?

Sketch of Proof
: downward smoothing

Key observation

$$\inf_{\rho' \in B^1_\epsilon(\rho), \rho \geq 0} D_{\alpha,Q}(\rho' \| \sigma) =: D^{\epsilon,\downarrow}_{\alpha,Q}(\rho \| \sigma)$$

is monotone decreasing by CPTP

Lower bund

$$D^{\epsilon,\downarrow}_{\alpha,Q}(\rho \| \sigma) \geq D^{\epsilon,\downarrow}_{\alpha,}(P^M_\rho \| P^M_\sigma)$$

RHS does not depends on Q

M: binary projection

Upper bund

$$D^{\epsilon,\downarrow}_{\alpha,Q}(\rho \| \sigma) \leq D^{\epsilon,\downarrow}_{\alpha}(\Gamma(p) \| \Gamma(q))$$

RHS does not depends on Q

Γ: CPTP $\quad p, q$: binary distributions

On upward smoothing

$$\sup_{\rho' \in B_\epsilon^1(\rho), \rho \geq 0} D_{\alpha,Q}(\rho' \| \sigma)$$

is NOT monotone decreasing by CPTP

You have to crack the problem one by one …

Summary

KL-divergence is stable both by downward- and upward smoothing

Renyi is not :

goes to divergence (rates), 0, ∞,

or log of smallest eigenvalue

downward :

not depend on which q-version

upward :

some dependency on q-version ?

Well...

When α∈(0,1), upward smoothing is not completely understood.

Here, we had considered constant smoothing.

$$\lim_{\varepsilon\downarrow 0} \overline{\lim_{n\to\infty}} \inf_{\rho'\in B^1_\varepsilon(\rho^n),\rho'\geq 0} D_{\alpha,Q}(\rho'\|\sigma^n)$$

How about exponentially vanishing smoothing ?

Generalization to GPT ?

Implication to Thermodynamics ?

Whenα