物性研究所 短期研究会
「量子情報・物性の新潮流　ー量子技術が生み出す多様な物性と情報処理理術ー」
2018年7月31日（火）ー8月3日（金）

ホログラフィック原理と情報幾何・エンタングルメント

松枝宏明
仙台高等専門学校

「量子系のエンタングルメントと幾何学
ホログラフィー原理に基づく異分野横断の数理」
森北出版（2016）

Outline

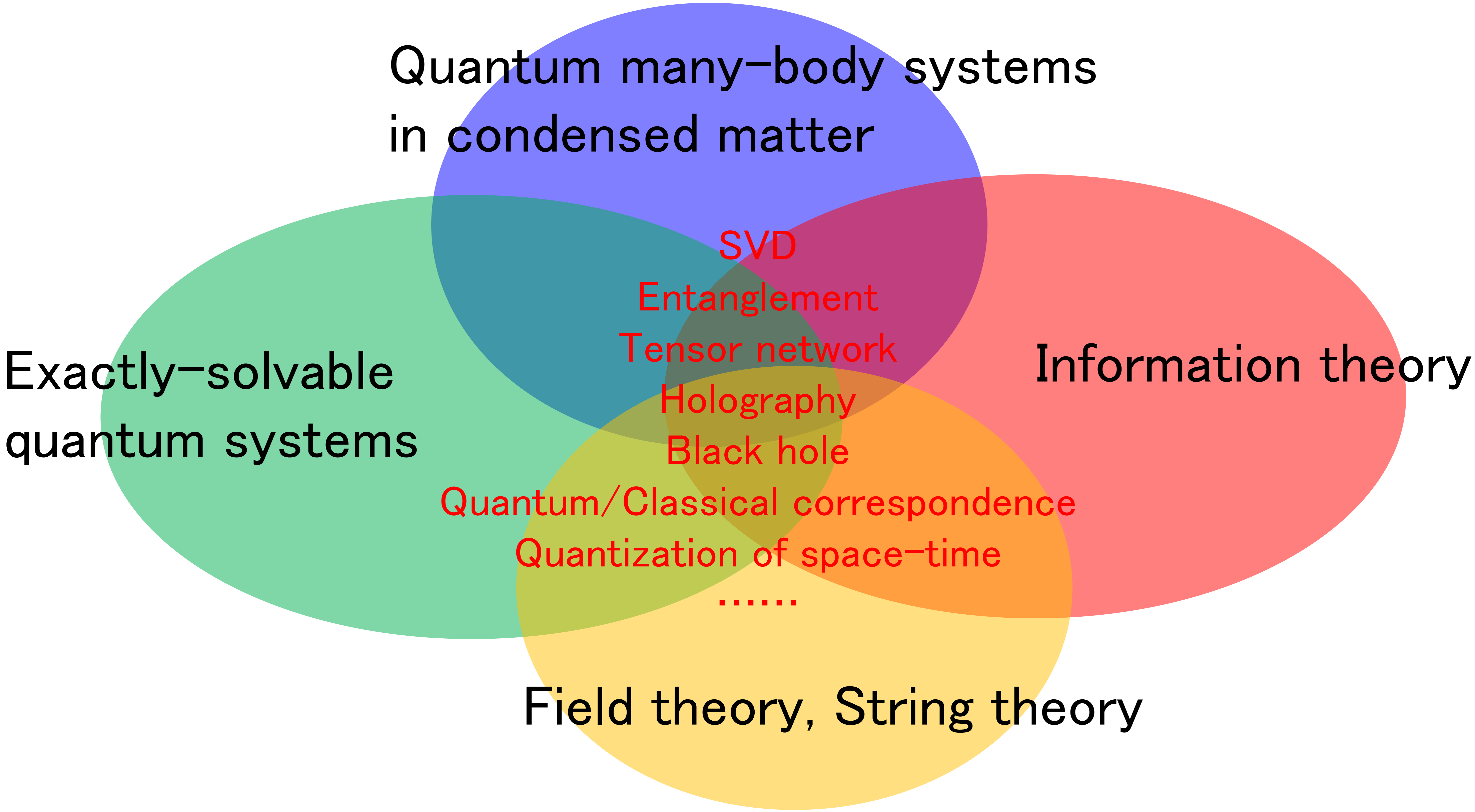
1. Introduction: Entanglement & Holography

2. Viewpoint of data storage in curved space-time:
   MERA tensor network, Extra dimension,
   and Holographic geometry

3. SVD is a good tool to understand holographic RG
   (although my works are still in a classical level ···)

4. Information-geometrical analysis of the correspondence
   between BTZ black hole and finite-T CFT

Recent development of interdisciplinary physics research



Key concepts

"quantum entanglement" and "holography principle"

<u>Entanglement entropy</u>

⇔SVD

Similar to the logarithm of two-point correlator

Scaling formula

⇔CFT

<u>Tensor network states</u>

Network geometry⇔RG

Variational ansatz for quantum many-body system which satisfies the entropy scaling

⇔Bethe ansatz

<u>Bulk/edge correspondence</u>

⇔wavelets

$AdS_{d+2} \Leftrightarrow CFT_{d+1}$

Classical side behaves as a memory to storage quantum data efficiently

⇔information geometry

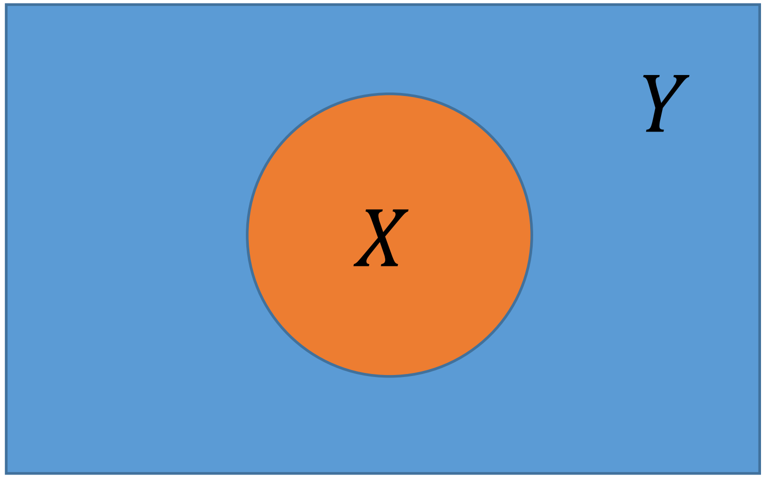
Reconstruction of statistical mechanics and field theory by the information-theoretical concepts

Introduction:
Entanglement and Holography

Entanglement entropy

total system (superblock, universe) = X+Y

$$|\psi\rangle = \sum_{x,y} \psi(x,y)|x\rangle \otimes |y\rangle \qquad x \in X, y \in Y$$



Reduced density matrices

$$\rho_X = \mathrm{Tr}_Y |\psi\rangle\langle\psi|$$

$$\rho_Y = \mathrm{Tr}_X |\psi\rangle\langle\psi|$$

Entanglement entropies

$$S_X = -\mathrm{Tr}_X(\rho_X \log \rho_X)$$

$$S_Y = -\mathrm{Tr}_Y(\rho_Y \log \rho_Y)$$

Entanglement entropy
⇔ Information (flow) across
the boundary of X and Y

Singular Value Decomposition (SVD)

SVD of rectangular matrix $\psi(x,y)=\sum_l U_l(x)\sqrt{\lambda_l}V_l(y)$

Reduced density matrices

$$\rho_X(x,x')=\sum_y \psi(x,y)\psi^*(x',y)=\sum_l U_l(x)\lambda_l U_l^*(x')$$

$$\rho_Y(y,y')=\sum_x \psi(x,y)\psi^*(x,y')=\sum_l V_l(y)\lambda_l V_l^*(y')$$

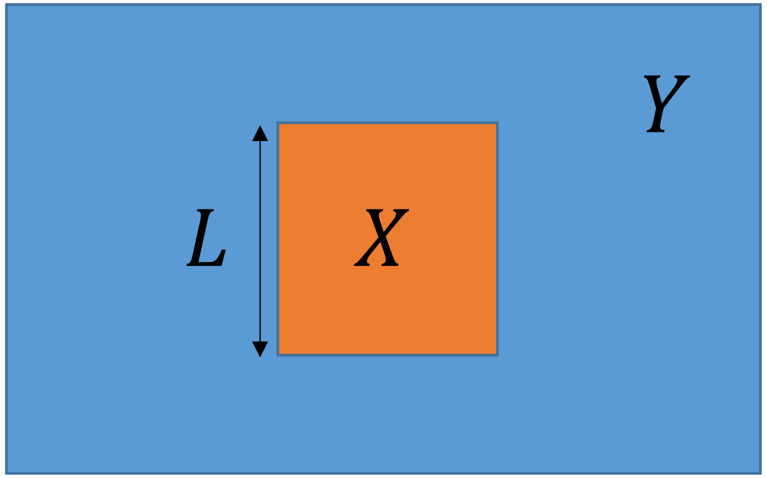
Entanglement entropy (von Neumann entropy in X or Y)

→ Violation of volume law scaling

$$S_X=-\sum_l \lambda_l \log \lambda_l = S_Y$$

Universal scaling formulae for entanglement entropy

Scaling formula depends on
(1) criticality, (2) space dimension d,
and (3) subsystem size (L: linear size)



$d = 2$

Gapped system → Area-law scaling

$S \propto L^{d-1}$ $\quad S \Leftrightarrow S_{BH}$ $\quad S_{th} \propto L^d$

Critical system (d=1) → logarithmic formula obtained by CFT

$$S(L) = \frac{c}{3}\log\left(\frac{N}{\pi}\sin\left(\frac{\pi L}{N}\right)\right) \to \frac{c}{3}\log L \qquad S = \frac{c}{3}\log\left(\frac{\beta}{\pi}\sinh\left(\frac{\pi L}{\beta}\right)\right)$$

$$S(N-L) = S(L)$$

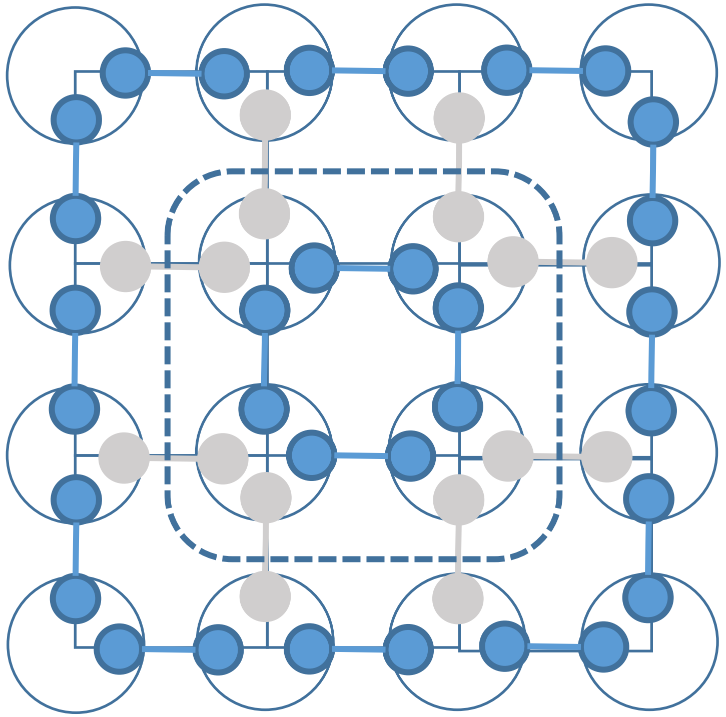
c: central charge of CFT

Application of the scaling to the construction of TNS

Tensor Network State (TNS), $d = 2$ $S = 4L \log \chi \propto L^{d-1}$

Projected Entangled Pair State(PEPS):

Variational ansatz that satisfies the correct area-law scaling



$$|\psi\rangle = \sum_{s_1, s_2, \cdots} \psi^{s_1 s_2 \cdots} |s_1 s_2 \cdots\rangle$$

Real spin configuration

$$\psi^{s_1 s_2 \cdots} = \sum A^{s_i}_{\alpha\beta\gamma\delta} B^{s_j}_{\delta\varepsilon\zeta\eta} \cdots$$

$$|\phi\rangle = \frac{1}{\sqrt{\chi}} \sum_{\alpha=1}^{\chi} |\alpha\rangle \otimes |\alpha\rangle$$

Hidden dimension → entanglement

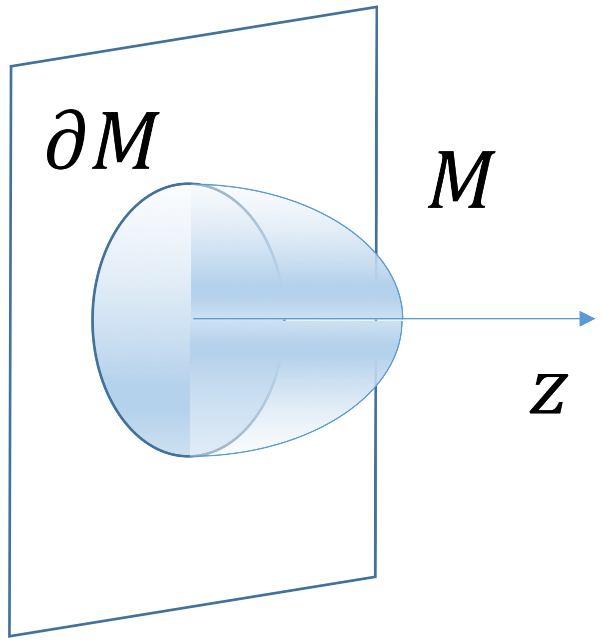
Holography principle

Holography principle

Classical gravity theory on a manifold M

(duality relation)

Quantum theory without gravity at ∂ M



AdS/CFT correspondence

Supersymmetric Yang–Mills theory ⇔ Type IIB string theory On $AdS_5 \times S^5$

(d+1)–dim. Quantum system with conformal symmetry

(d+2)–dim. Classical general relativity on anti–de Sitter space

Anti de Sitter (AdS) space

$$x^0 = t, x^1 = x, x^2 = z$$

AdS metric (d=1)

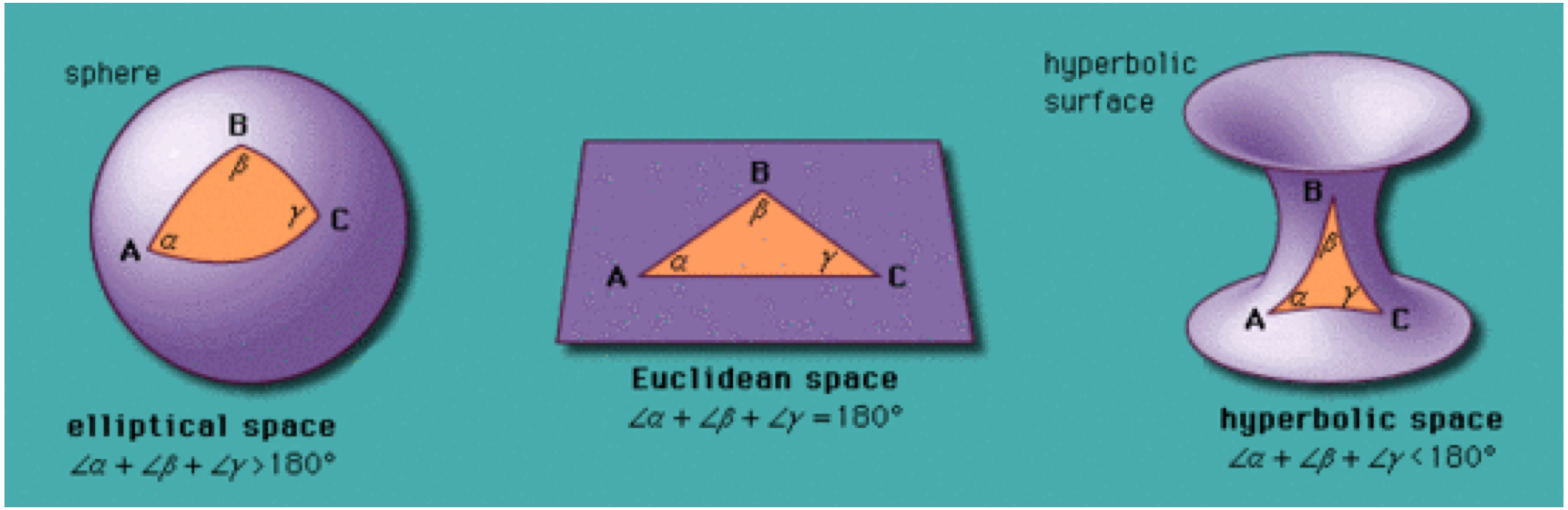
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{z^2}(-dt^2 + dz^2 + dx^2)$$

$$ds \sim \frac{l}{z} dz$$

$$s \sim l \log z$$

Hyperbolic geometry $ds^2 = \dfrac{dz^2 + dx^2}{z^2}$

l : curvature radius



CFT nature at the boundary of AdS

$$x^0 = t, x^1 = x, x^2 = z$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{l^2}{z^2}(-dt^2 + dz^2 + dx^2)$$

Coordinate transformation

$$\bar{x}^i = x^i + \xi^i(x) \qquad i = 0, 1$$
$$\bar{z} = z + z\zeta(x)$$

Scale invariance can be easily checked

Boundary of AdS_{d+2} ⇔ CFT_{d+1}

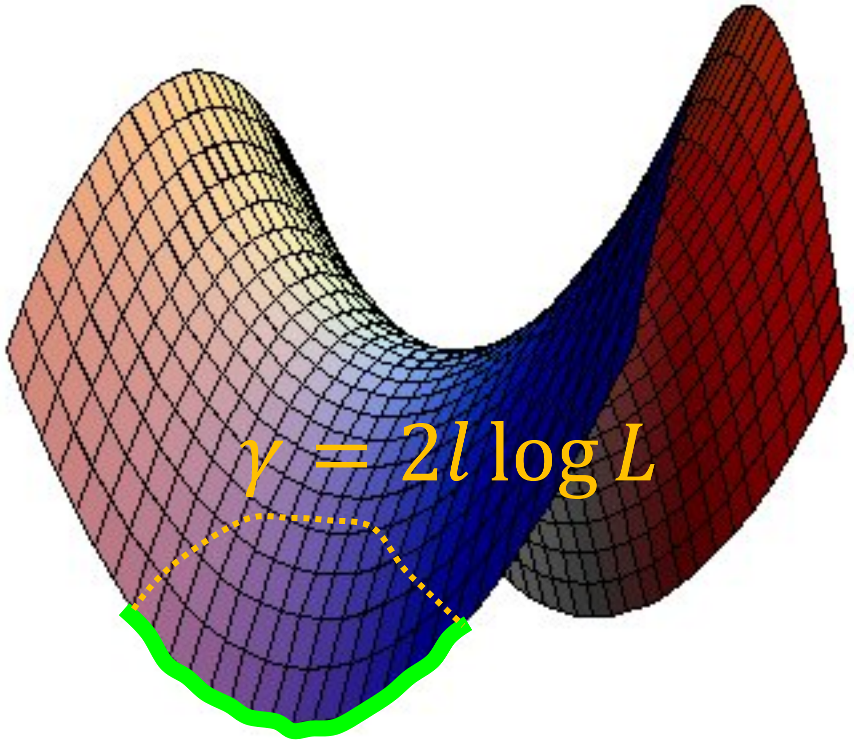
$$z \to 0$$

$$d\bar{s}^2 = ds^2 + (\partial_i \xi_j + \partial_j \xi_i - 2\zeta\eta_{ij}) dx^i dx^j$$

Isometry → conformal Killing equation at z=0

Holographic entanglement entropy

Entanglement⇔geometry



Ryu–Takayanagi formula (2006)

$$S = \frac{\gamma}{4G}$$

Extension of Beckenstein–Hawking black–hole entropy

$$S = \frac{\gamma}{4G} = \frac{l}{2G}\log L = \frac{c}{3}\log L$$

γ : Area of the minimal surface (d=1 → geodesic distance)

$$c = \frac{3l}{2G}$$ Brown–Henneaux central charge

Short summary:
What is a core algorithm of Quantum/Classical correspondence

Central issue: quantum/classical correspondence

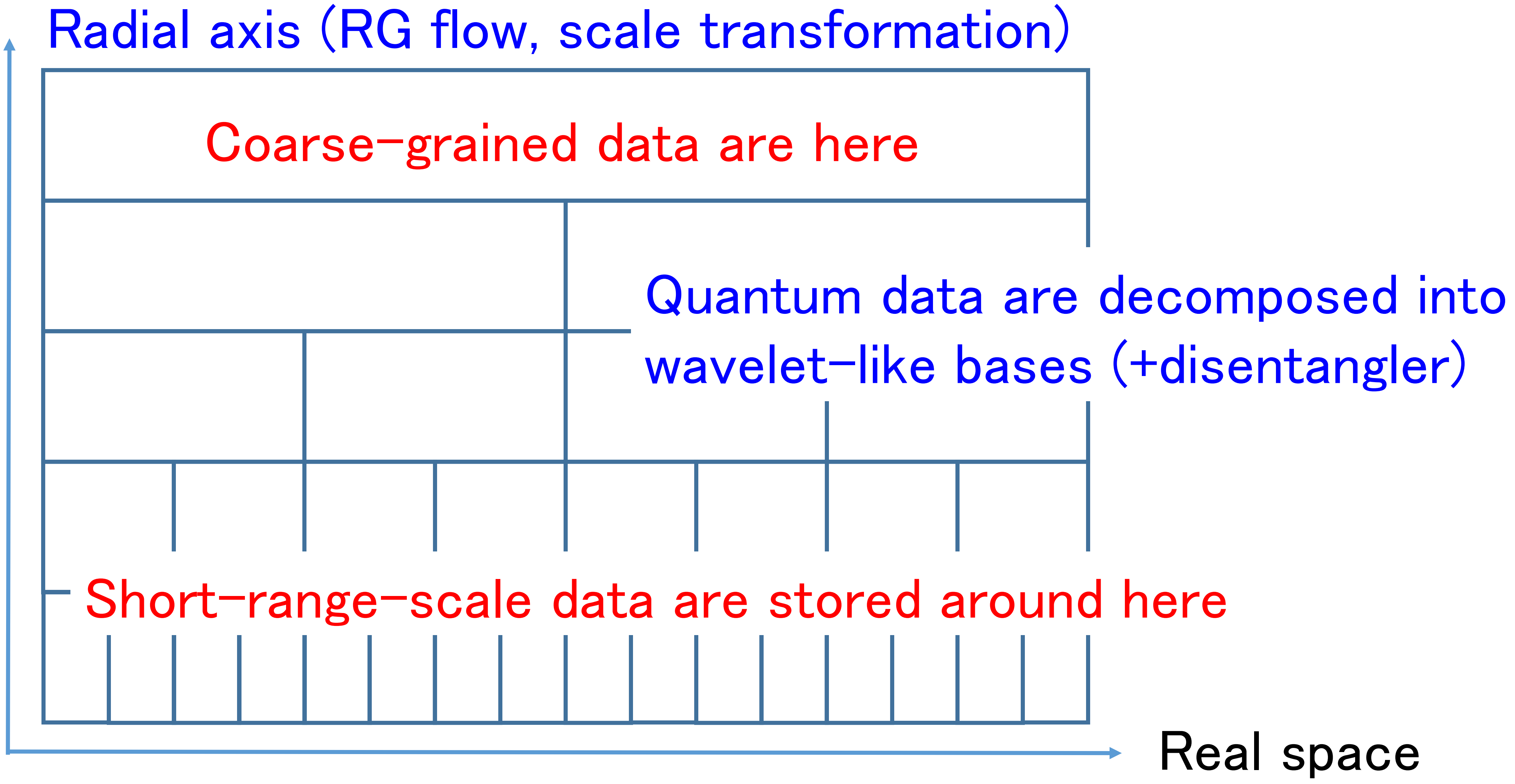
Entanglement (quantum side)

Dual object

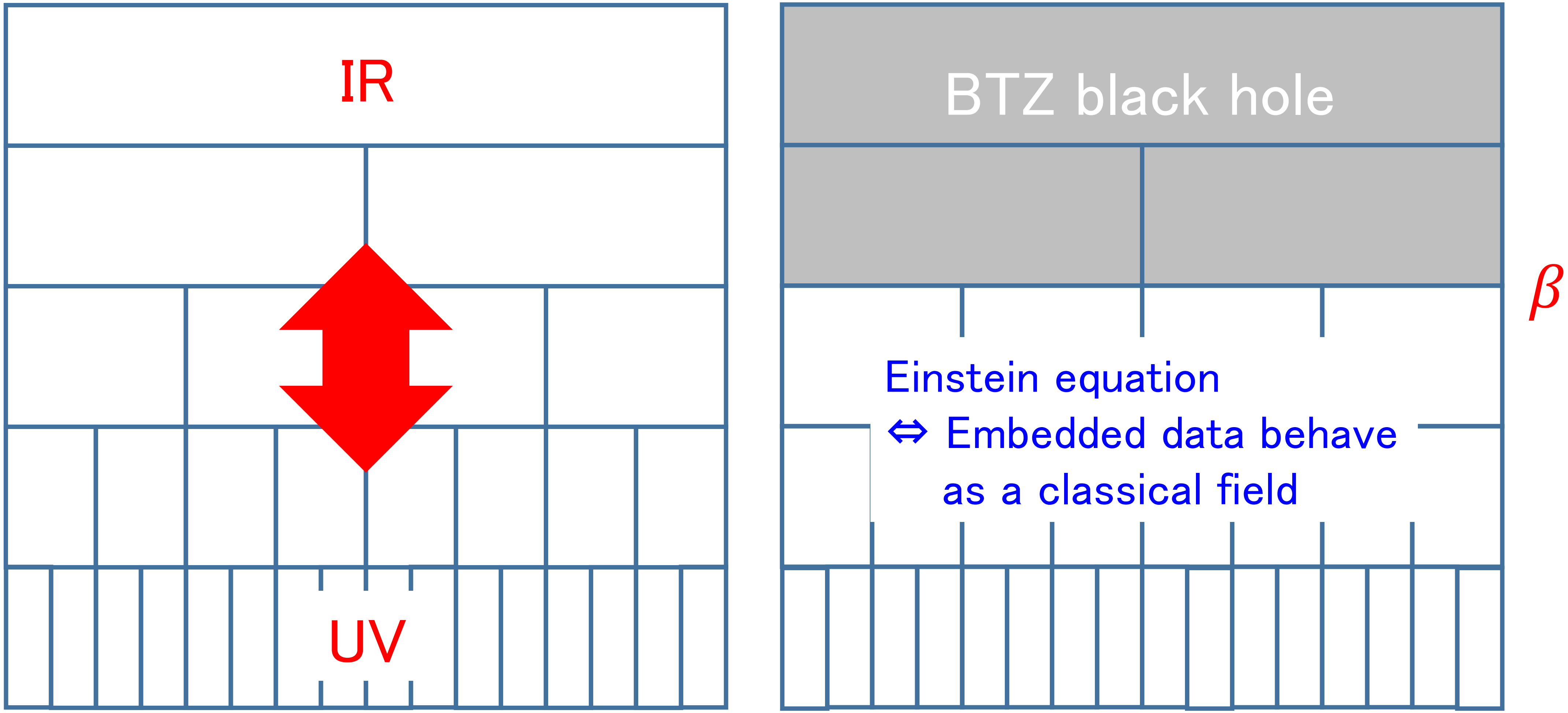
Geometry (classical side)

MERA tensor network, Extra dimension, and Holographic geometry

Quatum-data storage to hyperbolic space



Information-theoretical interpretation of AdS/CFT



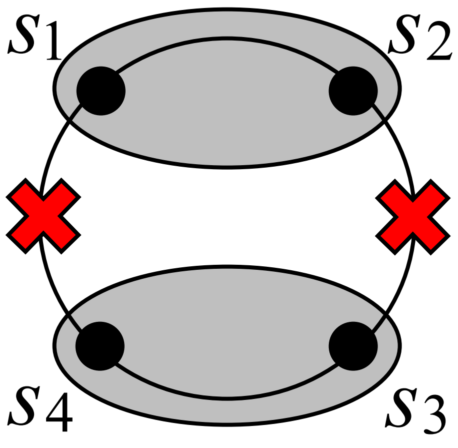
Ground state (critical)

Finite-T, excitation

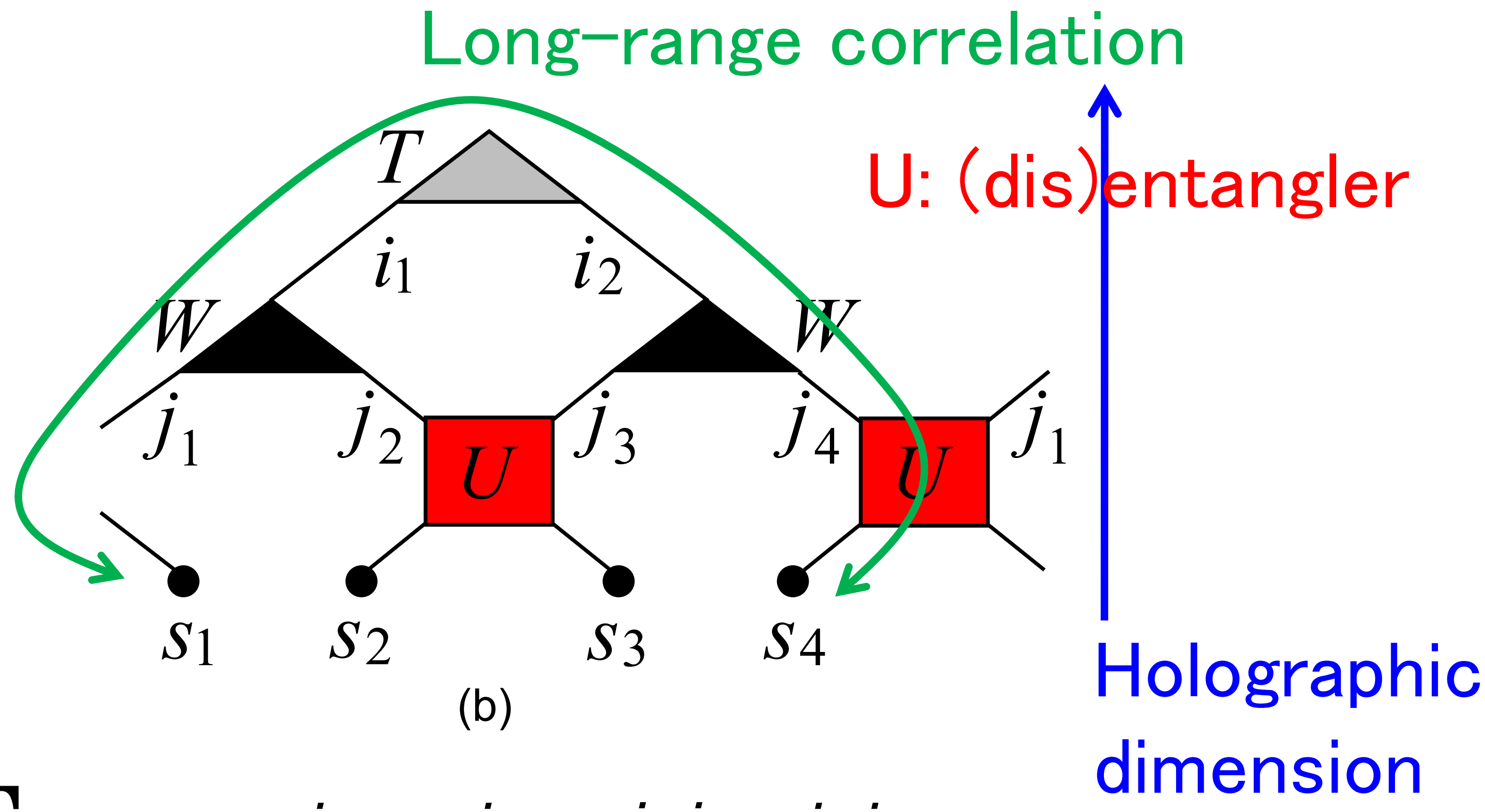
Hierarchical tensor network (entanglement renormalization)

Multiscale Entanglement Renormalization Ansatz (MERA)

$$H = J \sum_i \vec{S}_i \cdot \vec{S}_{i+1}$$



(a)

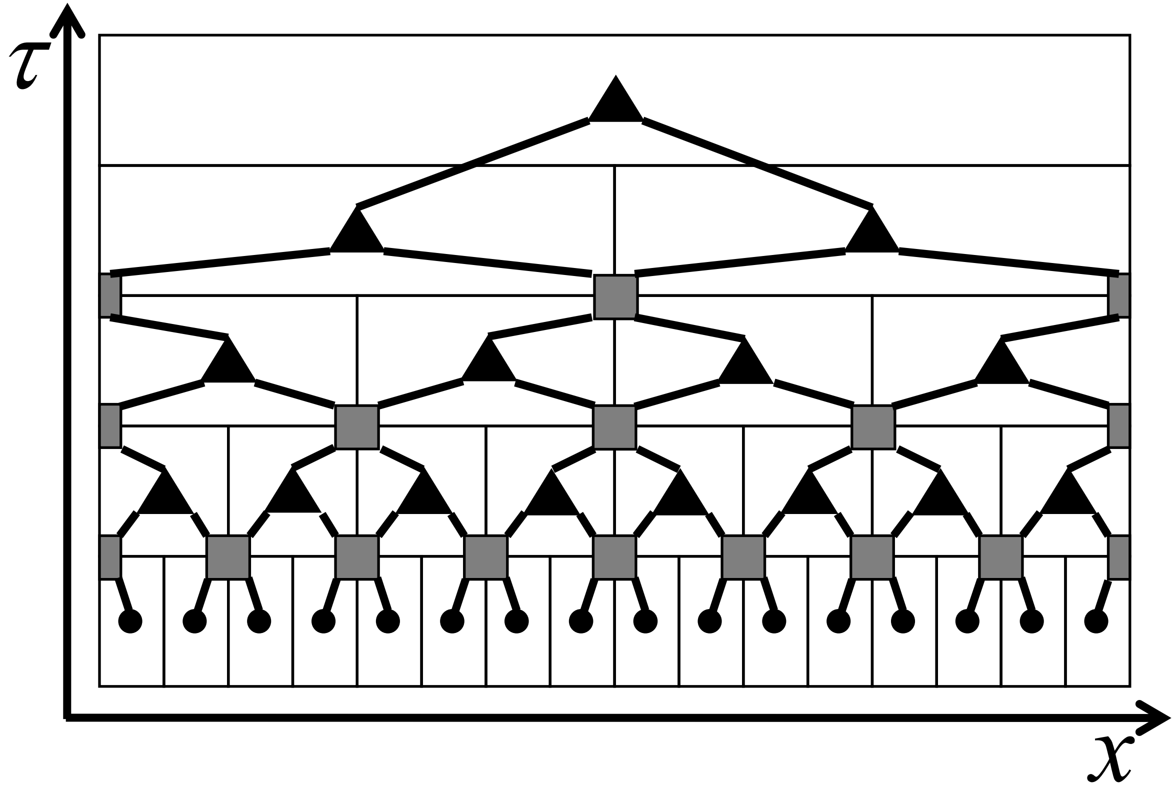


(b)

$$|\psi\rangle = \sum_{i_1,i_2} \sum_{j_1,\ldots,j_4} \sum_{s_1,\ldots,s_4} T_{i_1 i_2} W^{i_1}_{j_1 j_2} W^{i_2}_{j_3 j_4} U^{j_2 j_3}_{s_2 s_3} U^{j_4 j_1}_{s_4 s_1} |s_1 s_2 s_3 s_4\rangle$$

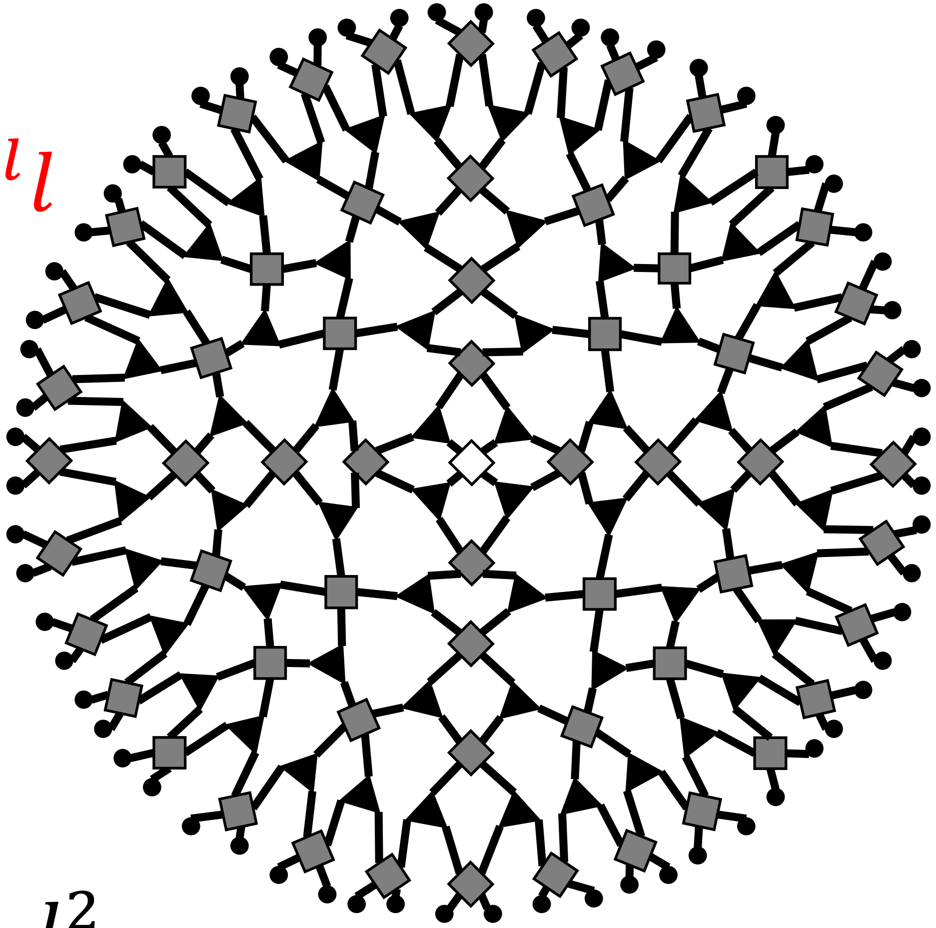
MERA network → discretized hyperbolic space

Upper-half of C



$z = 2^{\tau/l}l$

Poincare disk representation

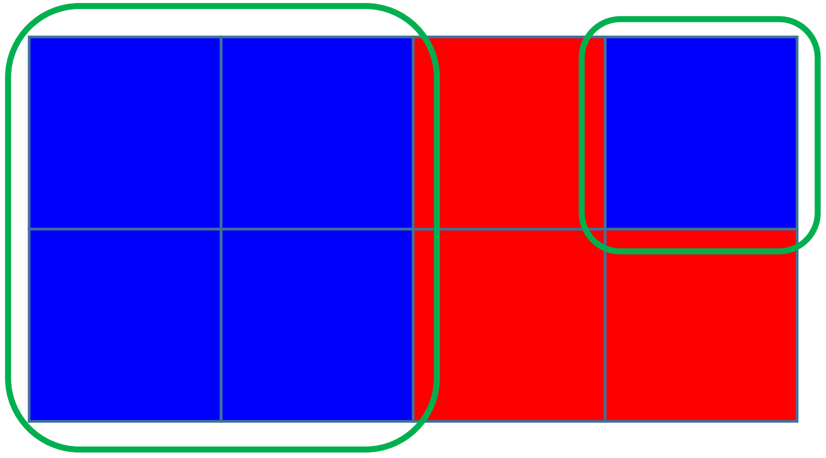


$$ds^2 = (d\tau \log 2)^2 + \left(2^{-\tau/l}dx\right)^2 = \frac{l^2}{z^2}(dz^2 + dx^2)$$

SVD is a good tool to understand holographic RG

C.H.Lee, D.Ozaki, HM, PRE94, 062144 (2016)
HM, JPSJ85, 114001 (2016)
HM, C,H,Lee, Y.Hashizume, JPSJ85, 086001 (2016)
HM, D.Ozaki, PRE92, 042167 (2015)
C.H.Lee, Y.Yamada, T.Kumamoto, and HM, JPSJ84, 013001 (2015)
HM, PRE85, 031101 (2012)

SVD and geometry with extra dimension (classical model)



$$M = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}$$

$$\Lambda_1 = 6, \Lambda_2 = 2$$

$$MM^T = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix} \qquad M^T M = \begin{pmatrix} 2 & 2 & -2 & 0 \\ 2 & 2 & -2 & 0 \\ -2 & -2 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

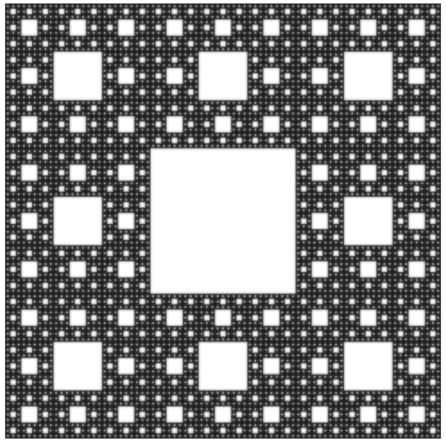
$$U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad V_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \quad V_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

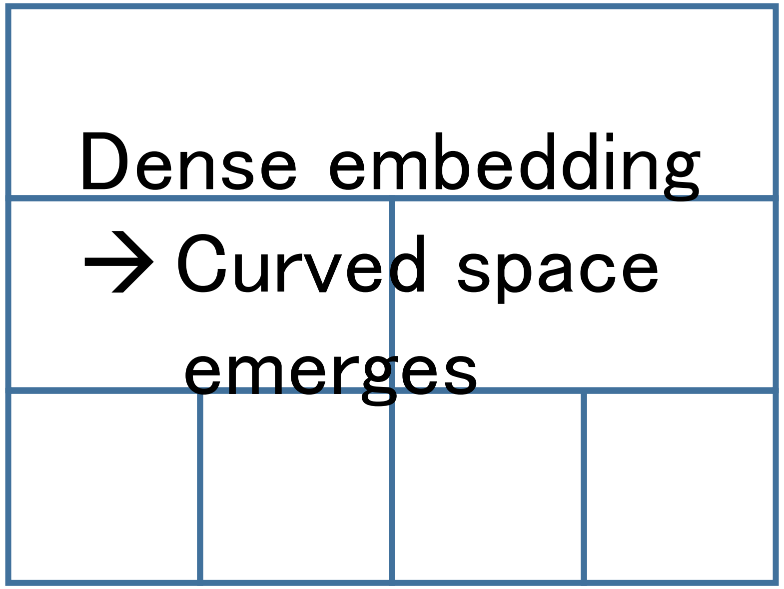
Large scale

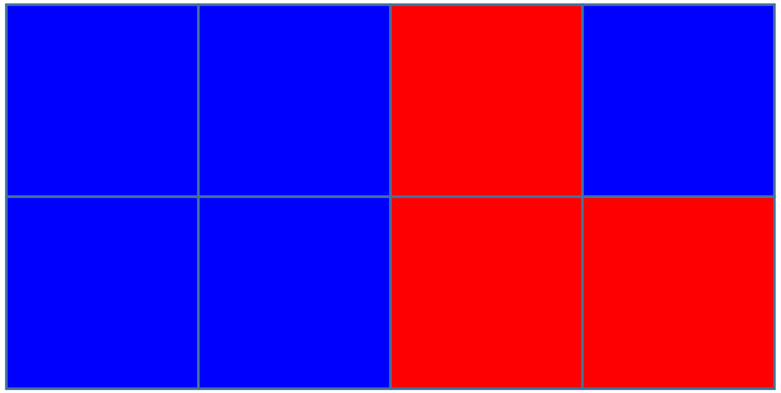
$$U_1\sqrt{\Lambda_1}V_1^T = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{pmatrix}$$

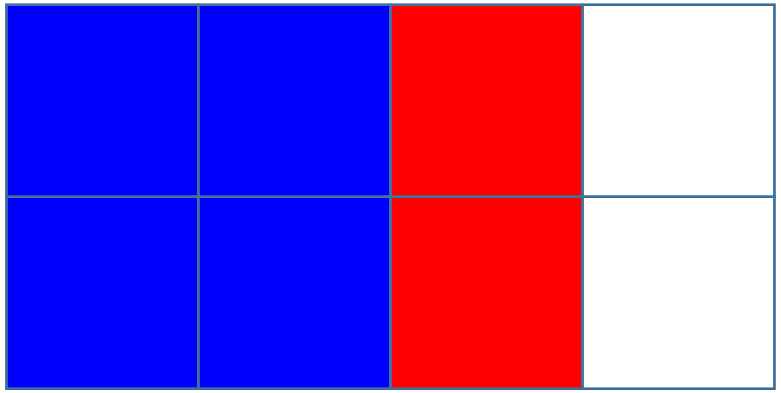
small scale

$$U_2\sqrt{\Lambda_2}V_2^T = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

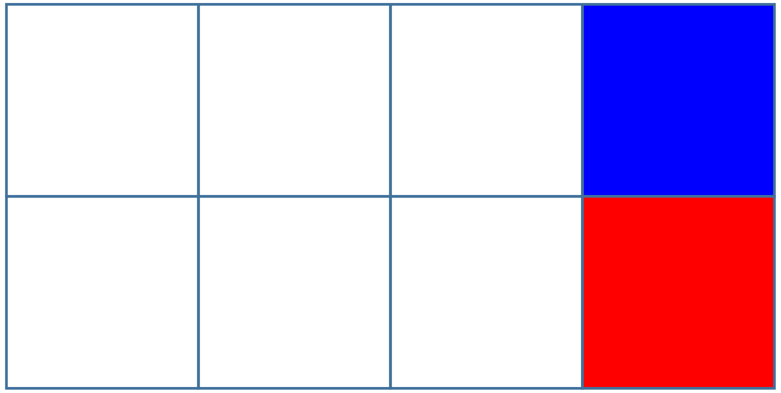








$\Lambda_1 = 6$

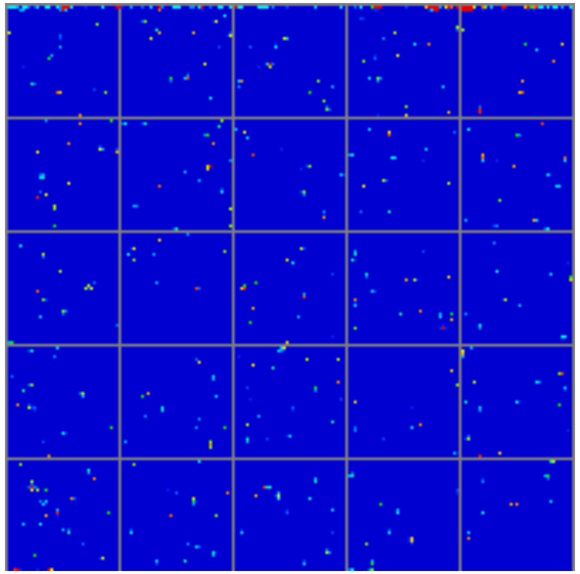


$\Lambda_2 = 2$

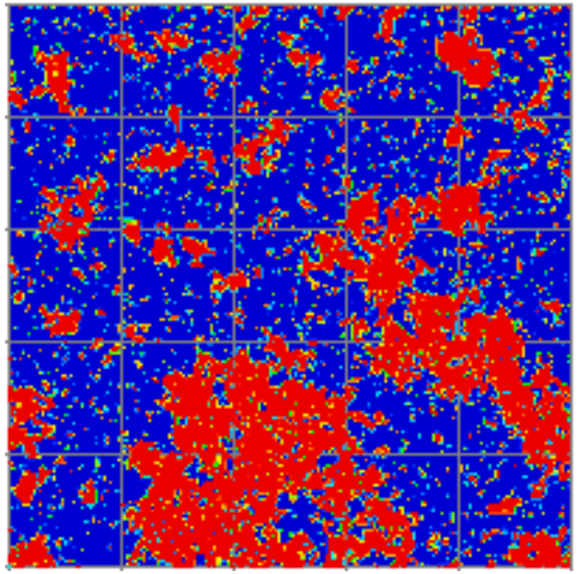
Monte Carlo Simulation of 2D Classical Ising Model

$$H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j \qquad \sigma_i = \pm 1 \qquad T_c = \frac{2J}{\log\left(1+\sqrt{2}\right)} = 2.269J$$

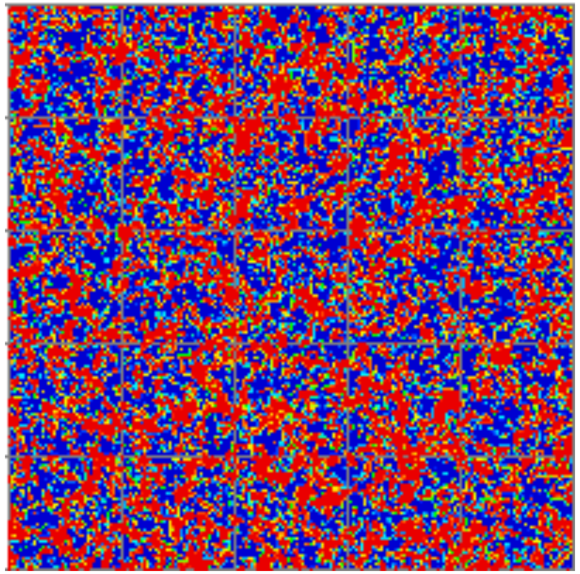
Typical snapshots at various temperatures → we can estimate T



$T = 1.52J$



$T = 2.27J \sim T_c$



$T = 3.02J$

$L = 256$

We need not to keep full information of partition function

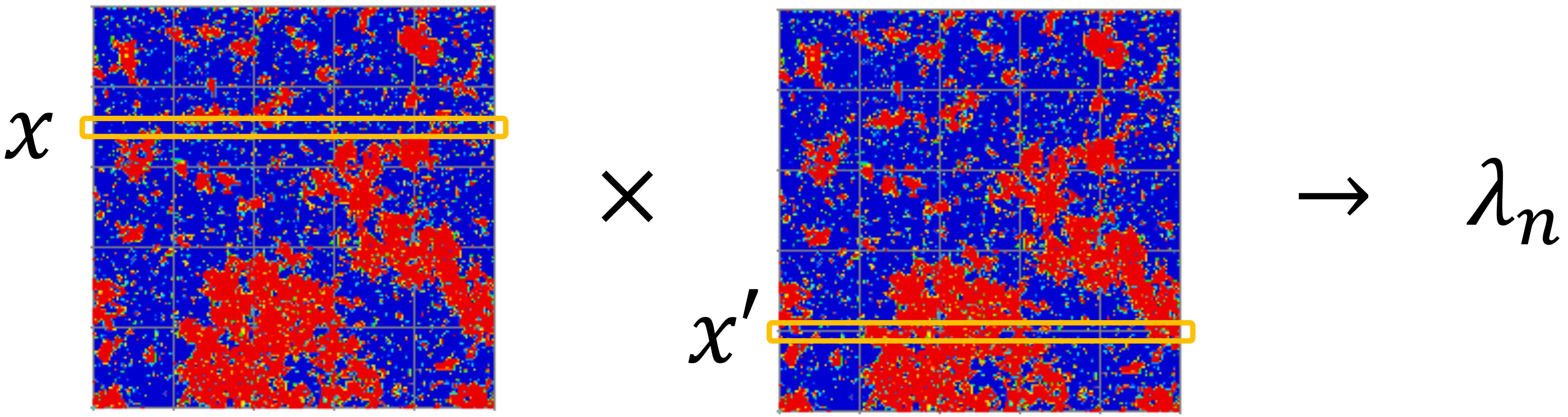
SVD = generator of 2-point correlator

Image-processing analysis of spin configuration by SVD
is closely related to the calculation of spin correlation function.

Typical processes of SVD

(1) Multiply M with M^T → spin correlation

(2) Diagonalization of MM^T and M^TM

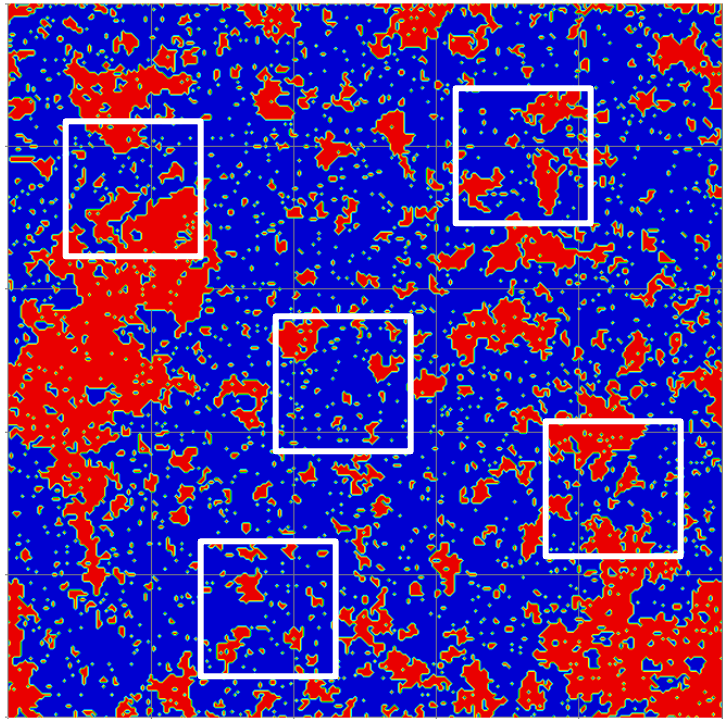


Full information of partition function would be extracted from single snapshot

Snapshot at Tc
→ Fractal-like spin configuration

typicality

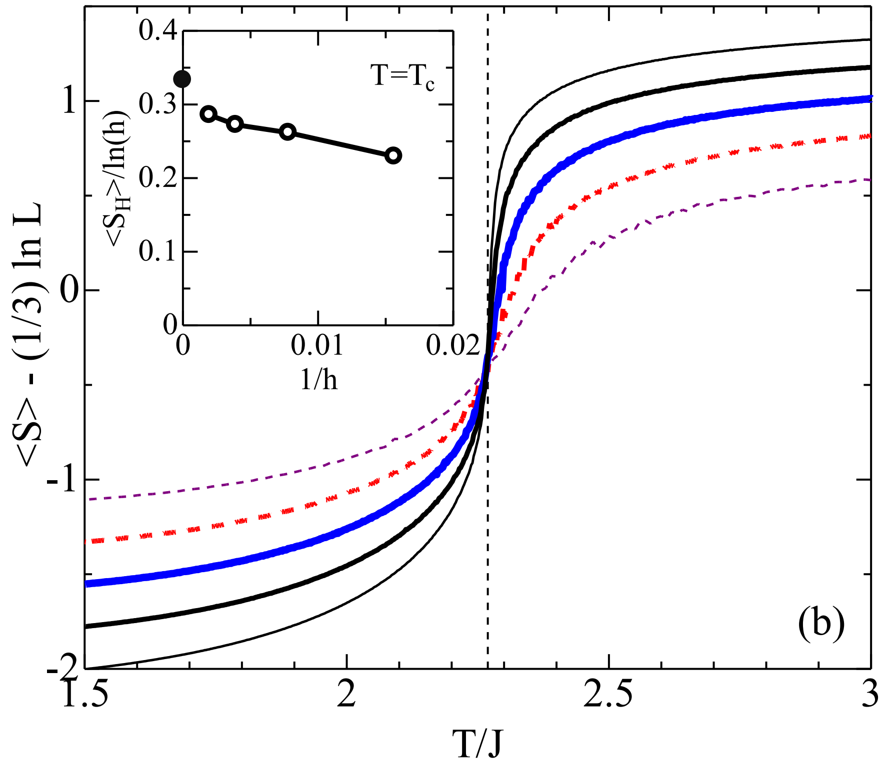
A set of partial systems roughly represents the information of all possible thermal fluctuation, if we neglect finite-size correction.

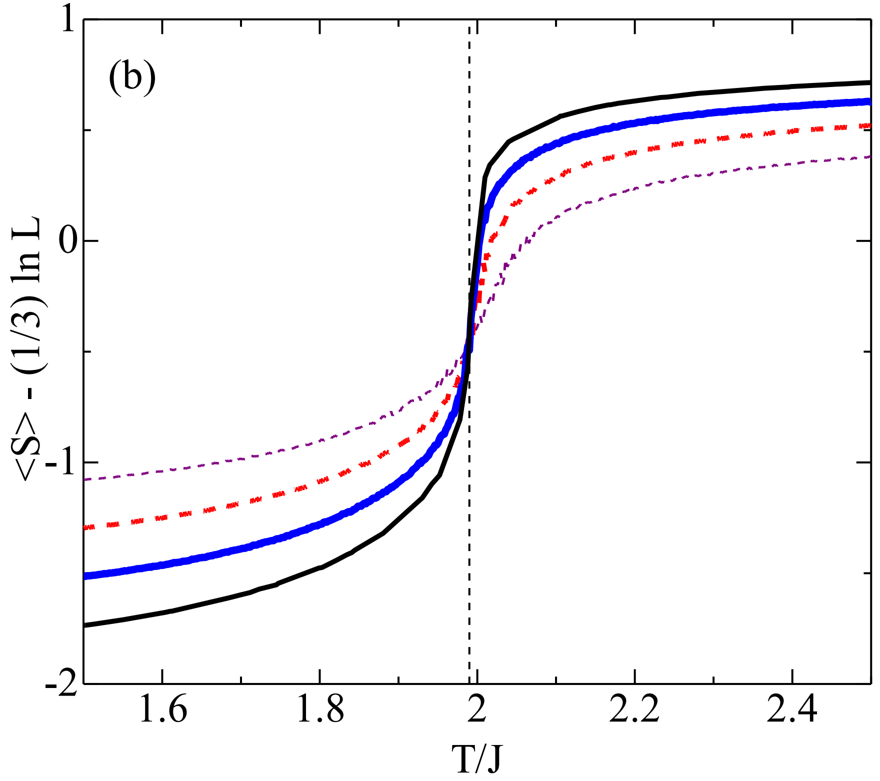


A typical snapshot at Tc
(256x256 sites)

Scaling formula for the snapshot entropy

(+1,−1) encoding



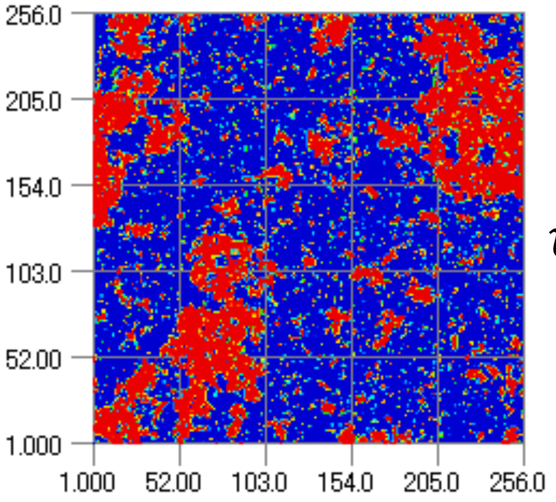


Ising model (c=1/2) 3-states Potts model (c=4/5)

$$S = \frac{1}{3}\log L - \frac{1}{2}$$

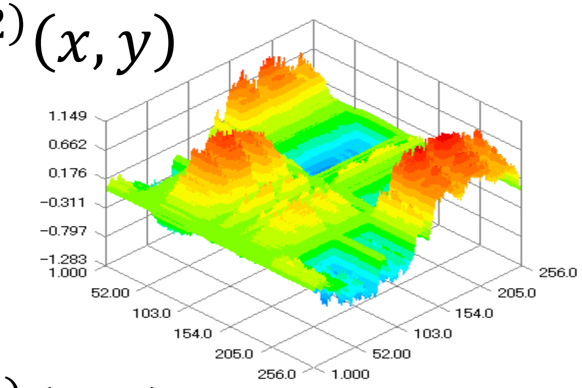
This scaling suggests the existence of a dual 1D quantum system

Hierarchical structure of SVD data

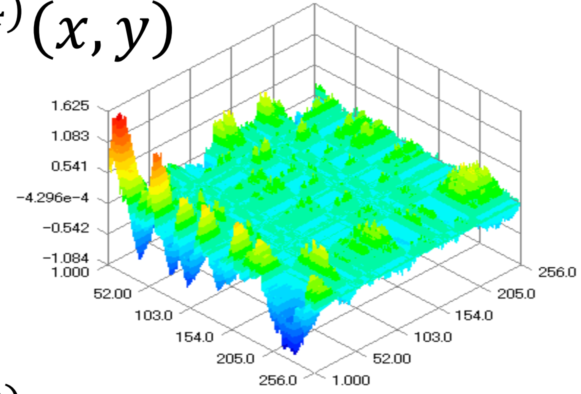


$\psi(x, y)$

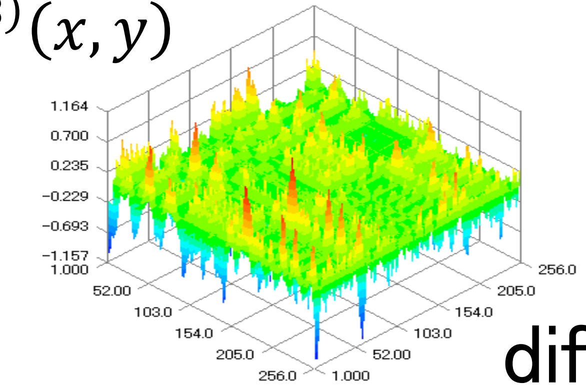
$\psi^{(2)}(x, y)$



$\psi^{(4)}(x, y)$



$\psi^{(8)}(x, y)$



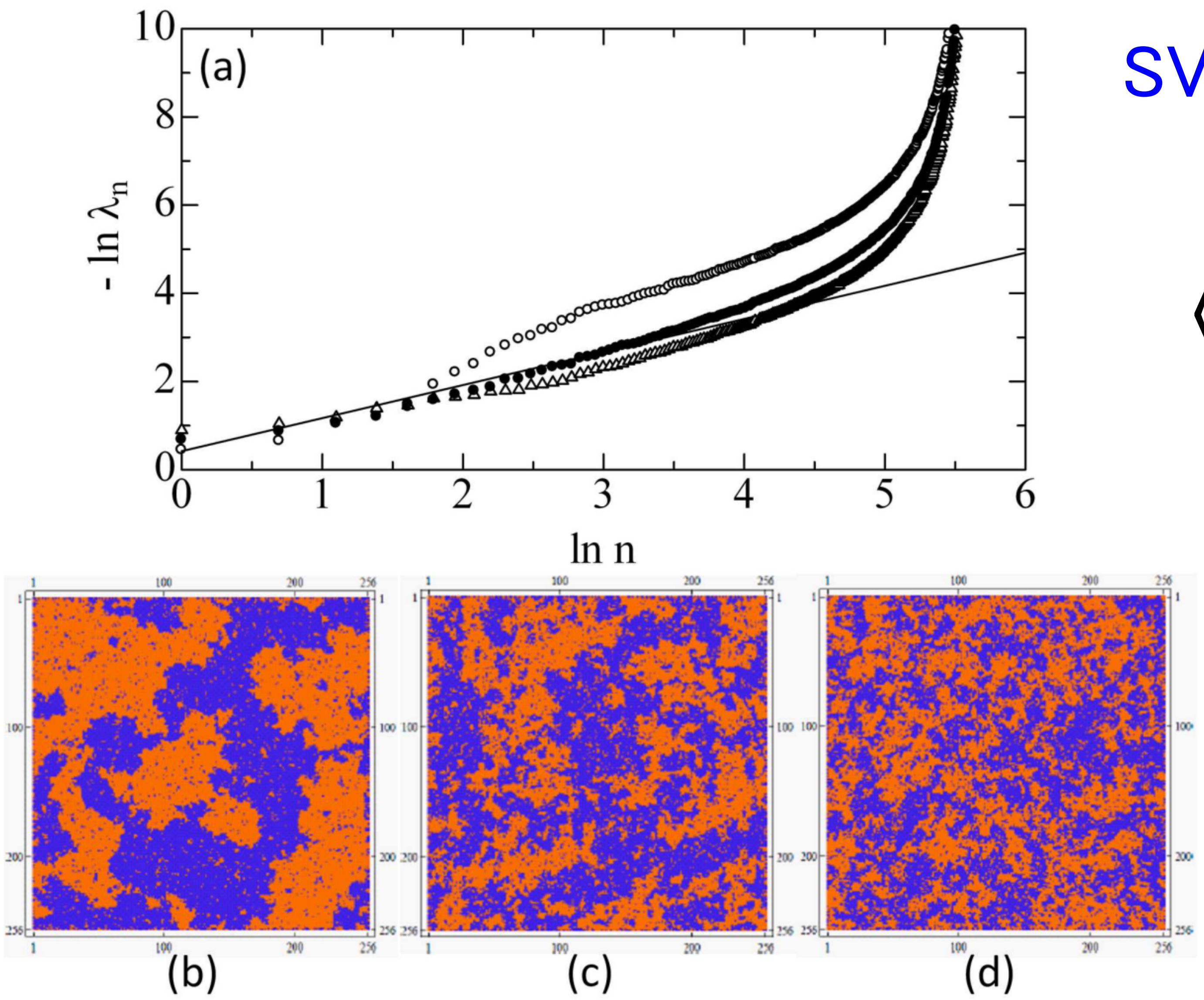
different scale

$$\psi(x, y) = \sum_l \psi^{(l)}(x, y)$$

$$\psi^{(l)}(x, y) = U_l(x)\sqrt{\Lambda_l}V_l(y)$$

Decomposition creates a holographic space where the length scale changes along l

What does SVD spectrum mean?



$$\eta = \frac{1}{4}$$

SVD → correlation function

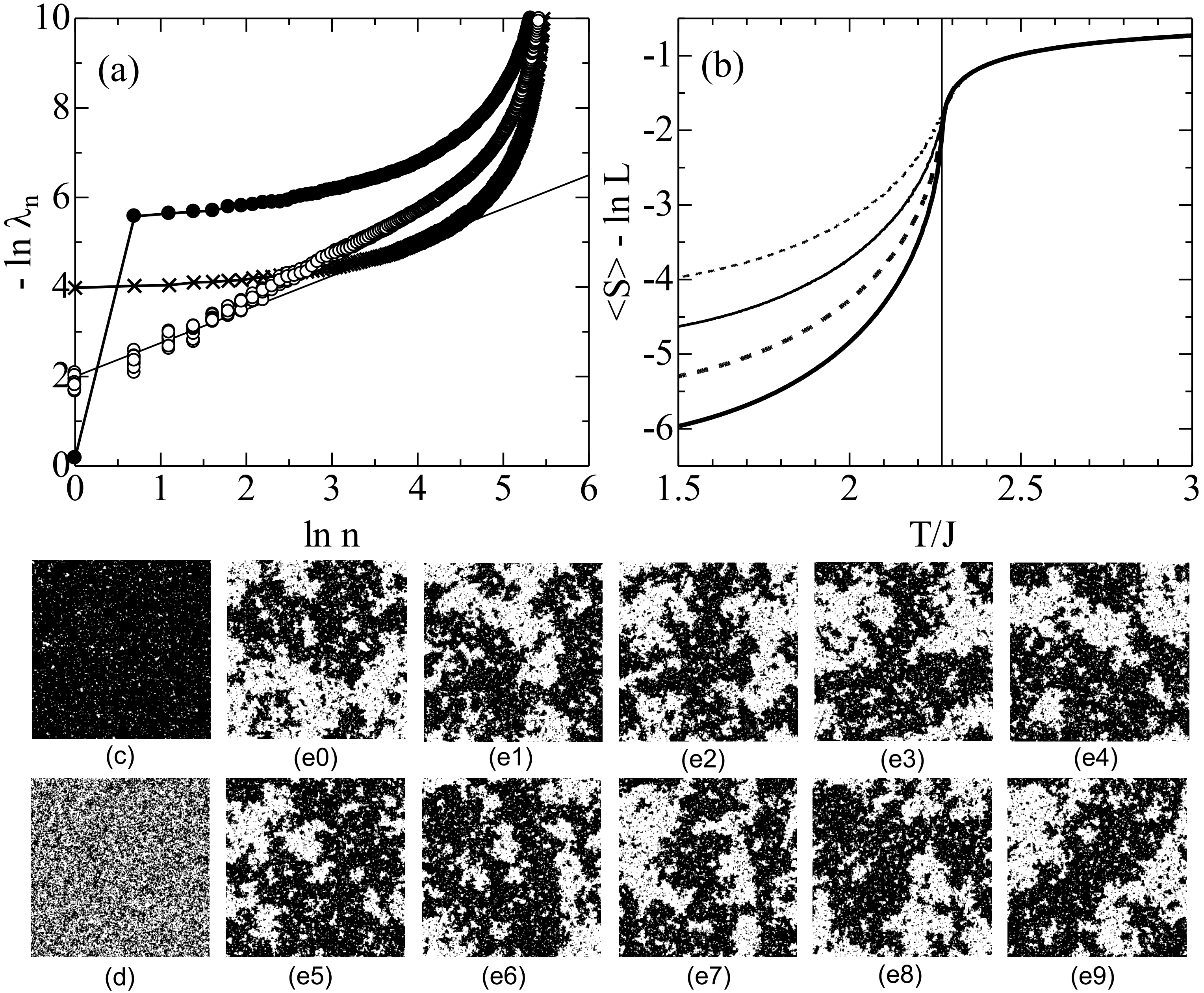
Two−point correlator

$$\langle \sigma(x)\sigma(y) \rangle \propto \frac{1}{|x-y|^{\eta}}$$

Hölder conjugate

SVD spectrum

$$\lambda_n \propto \frac{1}{n^{1-\eta}}$$



(a)
- ln λ_n
ln n
(b)
<S> - ln L
T/J
(c)
(e0)
(e1)
(e2)
(e3)
(e4)
(d)
(e5)
(e6)
(e7)
(e8)
(e9)

Continuous SVD for the Ising model at criticality

SVD of snapshot

$$M(x,y)=\sum_n U_n(x)\sqrt{\Lambda_n}V_n(y)$$

$\Lambda_n \propto n^{\eta-1}$

$d=2$

Reduced density matrix (= 2‒point correlator)

$$\rho(x,x')=\sum_y M(x,y)M(x',y)=\sum_n U_n(x)\Lambda_n U_n(x')$$

Decomposition formula

$$\frac{A}{|x-x'|^{d-2+\eta}}=\sum_n U_n(x)U_n(x')n^{\eta-1}=\sum_n R_n(x-x')n^{\eta-1}$$

$$\frac{A}{|x-x'|^\eta} = \sum_n R_n(x-x') n^{\eta-1} \qquad R_n(x-x') = U_n(x)U_n(x')$$

Decomposition formula in the continuous limit

→ Inverse Mellin transformation

$$\sum_n \cdots = \int dz \cdots$$

$$R(x-x', z) \propto \lim_{\eta' \to \eta - 0} \frac{e^{-z|x-x'|}}{(z|x-x'|)^{\eta'}} \qquad n \to z \propto \frac{1}{\xi}$$

Ornstein–Zernike form away from a critical point

Information-geometrical analysis of the correspondence between BTZ black hole and finite-T CFT

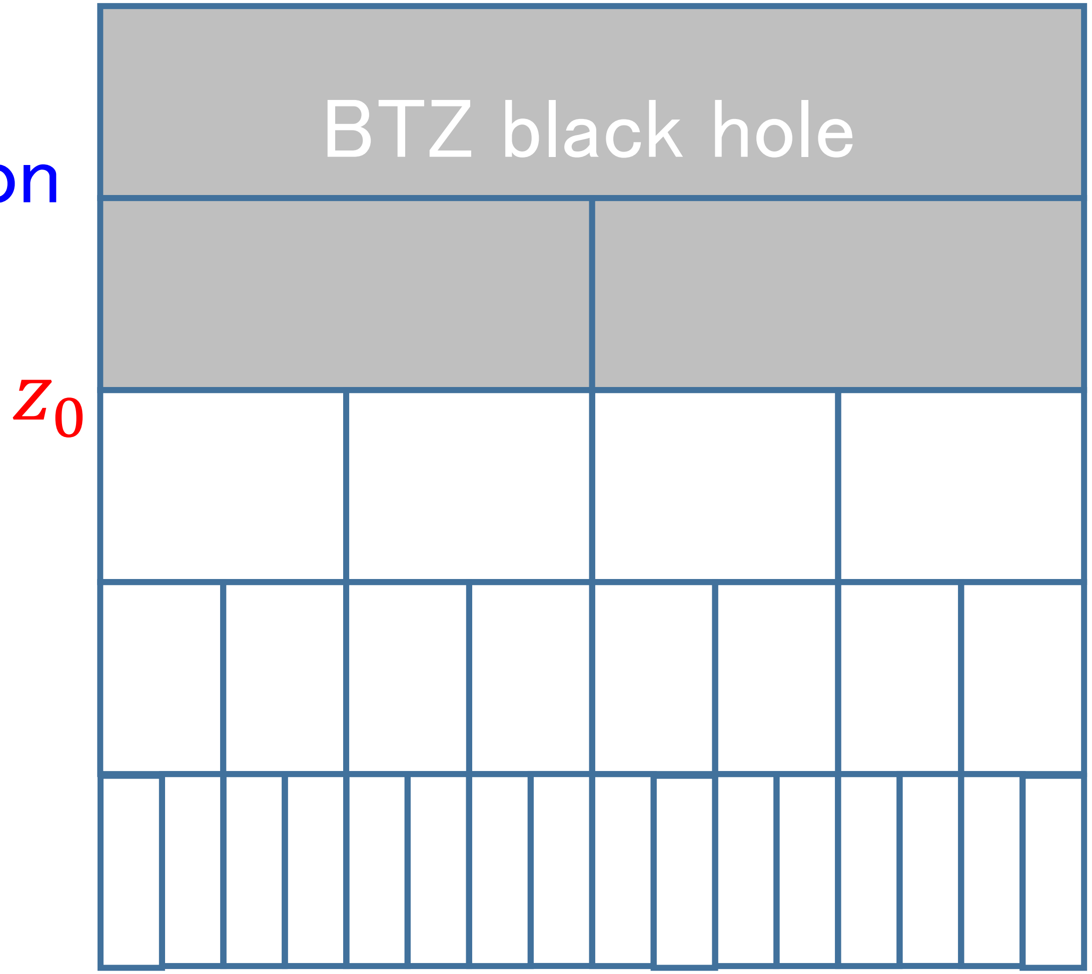
HM and T.Suzuki, JPSJ86, 104001 (2017)

BTZ black hole and holography

BTZ black hole (d=2): vacuum solution of Einstein equation with negative Λ

Dual QFT → finite-T CFT

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = 0$$



$$ds^2 = \frac{R^2}{z^2}\left\{-f(z)dt^2 + \frac{dz^2}{f(z)} + dx^2\right\}$$

$$f = 1 - az^2 = 1 - \left(\frac{z}{z_0}\right)^2 \qquad z_0 = \frac{\beta}{2\pi}$$

$$S = \frac{c}{3}\log\left(\frac{\beta}{\pi}\sinh\left(\frac{\pi L}{\beta}\right)\right)$$

Relative entanglement entropy

Schmidt decomposition of quantum pure state
(finite-T case is also represented by the same form using TFD)

$$|\psi(\theta)\rangle = \sum_n \sqrt{\lambda_n(\theta)} |n\rangle_A \otimes |n\rangle_{\bar{A}} \quad \langle\psi(\theta)|\psi(\theta)\rangle = \sum_n \lambda_n(\theta) = 1$$

θ : canonical parameters (function of model parameters)

Relative entropy and Fisher metric $\quad \gamma_n(\theta) = -\log\lambda_n(\theta)$

(this is a measure of difference between two states)

$$D(\theta) = \sum_n \lambda_n(\theta) \log\frac{\lambda_n(\theta + d\theta)}{\lambda_n(\theta)} = \frac{1}{2} g_{\mu\nu}(\theta) d\theta^\mu d\theta^\nu + \cdots$$

$$g_{\mu\nu}(\theta) = \sum_n \lambda_n(\theta) \frac{\partial \log\lambda_n(\theta)}{\partial\theta^\mu} \frac{\partial \log\lambda_n(\theta)}{\partial\theta^\nu} = \langle \partial_\mu\gamma \partial_\nu\gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle$$

Purpose of this study

$$|\psi(\theta)\rangle = \sum_n \sqrt{\lambda_n(\theta)}|n\rangle_A \otimes |n\rangle_{\bar{A}} \quad \langle\psi(\theta)|\psi(\theta)\rangle = \sum_n \lambda_n(\theta) = 1$$

$$\lambda_n(\theta)$$

Entanglement spectrum

$$\gamma_n(\theta) = -\log \lambda_n(\theta)$$

Entanglement entropy

$$S(\theta) = -\sum_n \lambda_n(\theta) \log \lambda_n(\theta) = \langle\gamma\rangle$$

Fisher metric

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \gamma \partial_\nu \gamma \rangle = \langle \partial_\mu \partial_\nu \gamma \rangle$$

To find the Schmidt coefficients that reproduce both CFT entanglement entropy and BTZ metric

Hessian Geometry and Entanglement Thermodynamics

$$|\psi(\theta)\rangle = \sum_n \sqrt{\lambda_n(\theta)}|n\rangle_A \otimes |n\rangle_{\bar{A}}$$

Exponential family (definition of canonical parameters θ)

Environment → finite-T effects

$$\psi(\theta) = \log Z$$

$$\lambda_n(\theta) = e^{-\gamma_n(\theta)} = \exp\{\theta^\mu F_{n,\mu} - \psi(\theta)\} = \frac{1}{Z} e^{\theta^\mu F_{n,\mu}}$$

Hessian geometry

$$\gamma_n(\theta) = \psi(\theta) - \theta^\mu F_{n,\mu}$$

Hessian potential

$$g_{\mu\nu}(\theta) = \langle \partial_\mu \partial_\nu \gamma \rangle = \partial_\mu \partial_\nu \psi(\theta)$$

Thermodynamics law for the entanglement entropy

$$S(\theta) = \langle \gamma(\theta) \rangle = \psi(\theta) - \theta^\mu \langle F_\mu \rangle = \psi(\theta) - \theta^\mu \partial_\mu \psi(\theta)$$

Geometry of Gaussian Distribution = Hyperbolic

$$p(X) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(X-\mu)^2}{2\sigma^2}\right\} = \exp\{\theta^1 F_1 + \theta^2 F_2 - \psi\}$$

$$\mu = \frac{\theta^1}{\theta^2}, \sigma = \frac{1}{\sqrt{\theta^2}} \qquad F_1 = X, F_2 = -\frac{1}{2}X^2$$

$$\psi = \log\left(\sqrt{2\pi}\sigma\right) + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 = -\frac{1}{2}\log\theta^2 + \frac{1}{2}\log 2\pi + \frac{1}{2}\frac{(\theta^1)^2}{\theta^2}$$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \left(\partial_\mu\partial_\nu\psi\right)dx^\mu dx^\nu = \frac{d\mu^2 + 2d\sigma^2}{\sigma^2}$$

Information-geometrical representation of BTZ black hole

$$\mu = \frac{\theta^1}{\theta^2}, \sigma = \frac{1}{\sqrt{\theta^2}} \Rightarrow t = \frac{\theta^0}{\theta^2 - a}, x = \frac{\theta^1}{\theta^2}, z = \frac{1}{\sqrt{\theta^2}}$$

$$t = \frac{\theta^0}{\theta^2 - a} = \frac{z^2}{1 - az^2}\theta^0 \Rightarrow \infty \ @z = z_0$$

Hessian potential that exactly produce BTZ metric

$$\psi = \frac{1}{4}\{(\theta^2 - a)\log(\theta^2 - a) - \theta^2 \log \theta^2\} + \frac{1}{2}\frac{(\theta^1)^2}{\theta^2} - \frac{1}{2}\frac{(\theta^0)^2}{\theta^2 - a}$$

$$S = \psi - \theta^\alpha \partial_\alpha \psi = \frac{1}{4}\log\left(\frac{z^2}{1 - az^2}\right) - \frac{1}{2}at^2$$

Close relationship with Ryu-Takayanagi formula

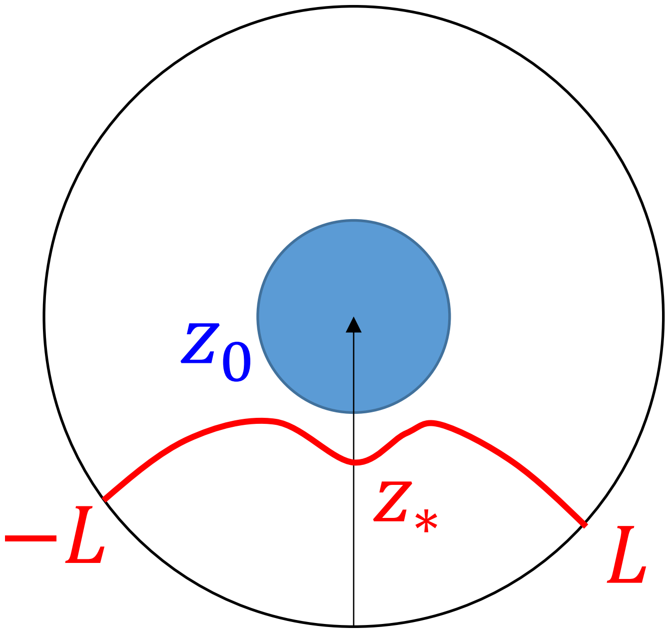
$$S(z,t) = \frac{1}{4}\log\left(\frac{z^2}{1-az^2}\right) - \frac{1}{2}at^2$$

$$z = z_* = z_0 \tanh\left(\frac{L}{z_0}\right) \qquad t = 0$$

$$S \Rightarrow \frac{1}{2}\log\left(z_0 \sinh\left(\frac{L}{z_0}\right)\right)$$

$$a = \left(\frac{1}{z_0}\right)^2 \qquad z = \frac{1}{\sqrt{\theta^2}}$$

Hawking radiation (?)



$$\theta^2 = \frac{1}{z^2} = \frac{1}{\left\{z_0 \tanh\left(\frac{L}{z_0}\right)\right\}^2} \Rightarrow \frac{1}{L^2}$$

θ^2 is a nontrivial function of L

Microscopic derivation from free Fermion entanglement

Hessian potential obtained here → phenomenological

$$\boxed{\theta^2 = \frac{1}{L^2}}$$

Reduced density matrix for free Fermion

(d=1, T=0, doping: δ)

$$\rho_A = \frac{1}{Z}\exp\left\{-\sum_{l=1}^{L}\varphi_l n_l\right\}$$

$$\varphi_l(L,\delta) = Lf(\delta,x)$$

$$x = \frac{l-l_F}{L} \qquad x = \delta L + \frac{1}{2}$$

$$f(\delta,x) = g(\delta)x^3$$ (numerical evidence)

$$\Delta = \theta^\mu\left(F_{1,\mu} - F_{2,\mu}\right) = \gamma_2 - \gamma_1 = \left\{\sum_{l=1}^{l_F-1}\varphi_l(L,\delta) + \varphi_{l_F+1}(L,\delta)\right\} - \sum_{l=1}^{l_F}\varphi_l(L,\delta)$$

$$= \varphi_{l_F+1}(L,\delta) - \varphi_{l_F}(L,\delta) = Lf\left(\delta,\frac{1}{L}\right) = \frac{g(\delta)}{L^2}$$

Summary

Reconstruction of statistical mechanics and field theory by using information-theoretical concepts

Information theory ⇔ CFT, integrability, geometry, ⋯

(1) Entanglement entropy scaling
(2) Extra dimension & tensor networks
(3) Network structure + RG concept → discretized geometry
(4) Rich physical aspects inherent in SVD
(5) information-geometrical interpretation of AdS/CFT