量子情報基礎

物性研短期研究会 量子情報･物性の新潮流 2018/7/31

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1. Basic rules of quantum mechanics
2. Density operators
3. Generalized measurements and quantum operations
4. Communication resources

Overview of the tutorial

Important technical tools

Properties of bipartite pure states

- Schmidt decomposition
- Local convertibility
- Relative states
- Bell basis

Basic rules

- Vectors
- Orthogonal measurements
- Unitary transformations

⇩

The most general descriptions

- Density operators
- Generalized measurements
- Quantum operations

Measure of distinguishability

- Fidelity
- No cloning theorem

Communication resources

- Classical channels
- Quantum channels
- Entanglement
- Entanglement sharing
- Quantum dense coding
- Quantum teleportation

Distinction from classical theory

- Partially distinguishable pair of pure states
- Mixed states are inevitable (entanglement)
- Looks as if the state could be chosen retroactively

1. Basic rules of quantum mechanics

Basic rule I: States

Basic rule II: Transformations

Basic rule III: Measurements

Basic rule IV: Compositions

Basic rule V: Causality

Basic rule I: States

A physical system is associated with a Hilbert space $\mathcal{H}$

Every pure state is represented by a normalized vector $|\phi\rangle \in \mathcal{H}$

For any normalized vector $|\phi\rangle \in \mathcal{H}$, it is possible to prepare the system in the state represented by $|\phi\rangle$

(Remarks)

We can also prepare the system in a mixed state according to an instruction $\{(p_j, |\phi_j\rangle)\}_j$
(probability, state)

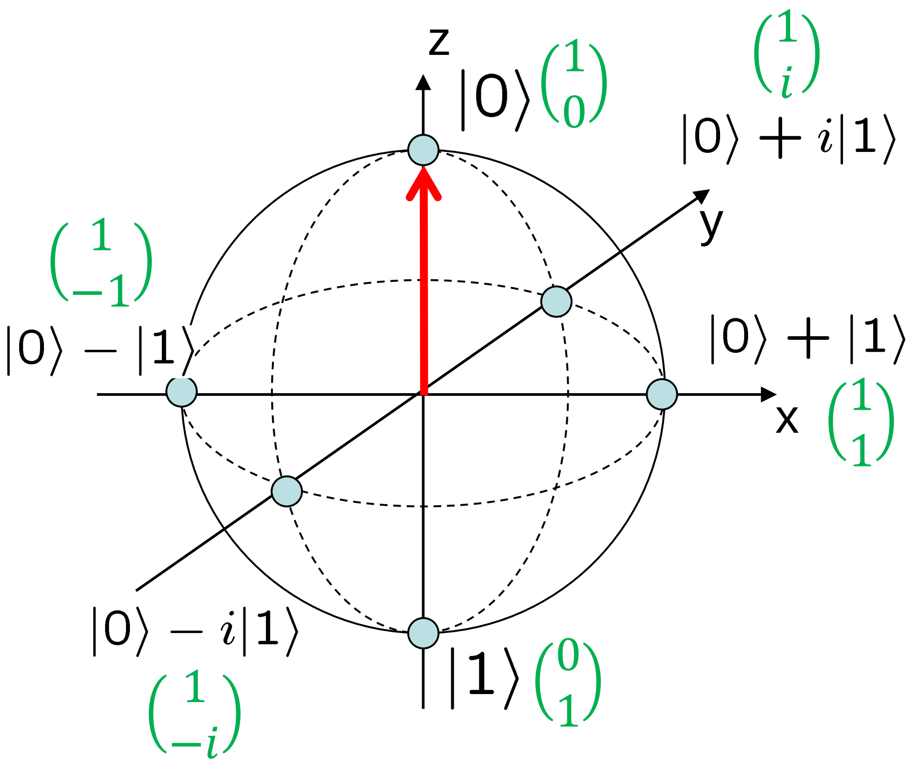
Hilbert space = vector space + inner product + completeness

A pair of pure states are either
$|\phi\rangle, |\psi\rangle \in \mathcal{H}$

- perfectly distinguishable $|\langle\psi|\phi\rangle| = 0$
- partially distinguishable $0 < |\langle\psi|\phi\rangle| < 1$
- completely indistinguishable $|\langle\psi|\phi\rangle| = 1$
  (the same physical states)

The representation is not 1-to-1. $e^{i\theta}|\phi\rangle$

Pure states and vectors



Spin-1/2 particle

(Coefficients $1/\sqrt{2}$ are omitted)

$|\phi\rangle \leftrightarrow \begin{pmatrix}a_1\\a_2\end{pmatrix}$ Dimension $\dim \mathcal{H} = 2$

'ket'

$\langle\phi| \leftrightarrow \begin{pmatrix}\overline{a_1} & \overline{a_2}\end{pmatrix}$

'bra'

$$\langle\phi|\phi\rangle = \begin{pmatrix}\overline{a_1} & \overline{a_2}\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix} = |a_1|^2 + |a_2|^2$$

$$|\phi\rangle\langle\phi| \leftrightarrow \begin{pmatrix}a_1\\a_2\end{pmatrix}\begin{pmatrix}\overline{a_1} & \overline{a_2}\end{pmatrix} = \begin{pmatrix}\blacksquare & \blacksquare\\ \blacksquare & \blacksquare\end{pmatrix}$$

Basic rule II: Transformations

For any unitary operator $\hat{U}$ on $\mathcal{H}$, it is possible to implement a state transformation

$$|\phi_{\rm out}\rangle = \hat{U}|\phi_{\rm in}\rangle$$

(Remarks)

Unitary operations are reversible. $\hat{U}^{-1} = \hat{U}^\dagger$

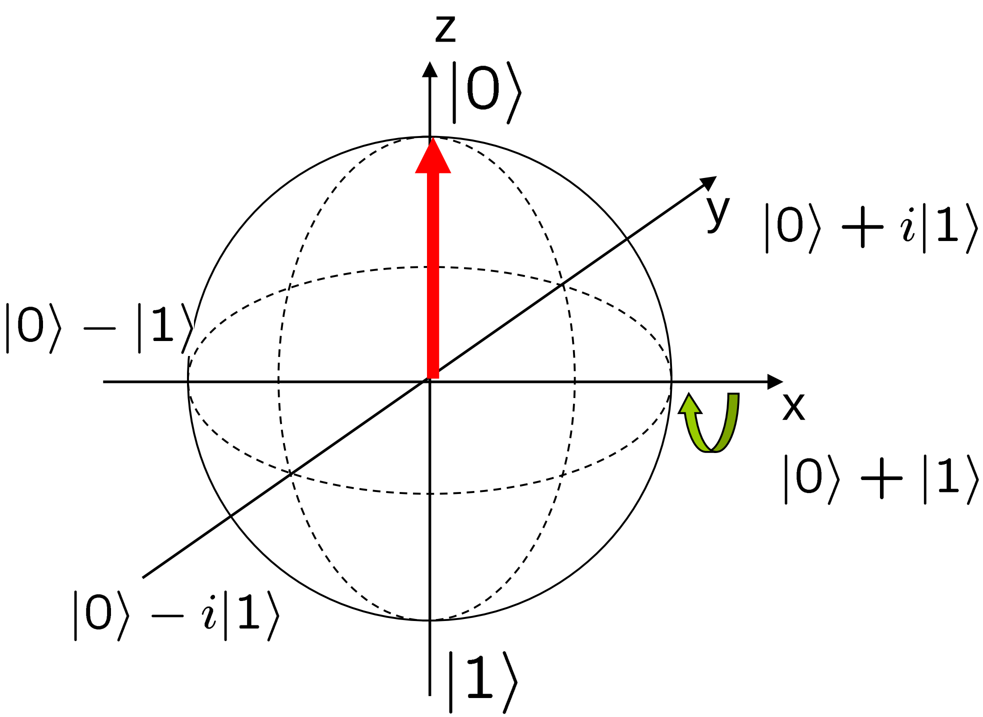
Inner products are preserved by unitary operations. $\langle\psi|\phi\rangle = \langle\psi|\hat{U}^\dagger\hat{U}|\phi\rangle$

Infinitesimal change $|\phi(t+dt)\rangle = \hat{U}(t+dt,t)|\phi(t)\rangle$

$$\downarrow \quad \hat{U}(t+dt,t) \cong \hat{1} - (i/\hbar)\hat{H}(t)dt$$

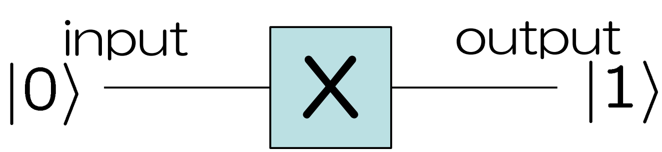
Schrödinger equation $$i\hbar\frac{d}{dt}|\phi(t)\rangle = \hat{H}(t)|\phi(t)\rangle$$

Unitary operations



$$|\phi_{\rm out}\rangle = \hat{U}|\phi_{\rm in}\rangle$$

$$\begin{pmatrix}\beta\\ \alpha\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$



X gate

Basic rule III: Measurements

For any orthonormal basis $\{|u_j\rangle\}_{j=1,\ldots,d}$ of $\mathcal{H}$, it is possible to implement a measurement that produces an outcome $j = 1, \ldots, d$ with

$$\Pr\{j\} = |\langle u_j|\phi_{\text{in}}\rangle|^2$$

(Remarks)

$d = \dim \mathcal{H}$ is the maximum number of mutually distinguishable states (d-level system)

Measurement of an observable

Self-adjoint operator $\hat{A} = \sum_j \lambda_j |a_j\rangle\langle a_j|$

$\downarrow$

Measurement on the basis $\{|a_j\rangle\}_{j=1,\cdots,d}$ Assign $j \to \lambda_j$

Expectation value

$$\langle \hat{A} \rangle \equiv \sum_j P(j)\lambda_j = \sum_j \langle \phi|a_j\rangle\langle a_j|\phi\rangle\lambda_j = \langle \phi|\hat{A}|\phi\rangle$$

Basic rule IV: Compositions

basis

$\{|a_i\rangle\}_{i=1,\ldots,d}$

$\mathcal{H}_A$

System A

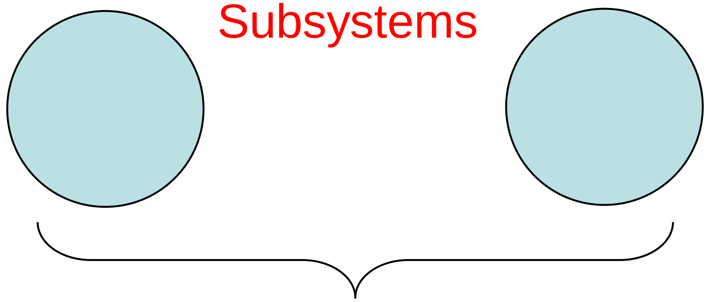
$\mathcal{H}_B$

System B

basis

$\{|b_j\rangle\}_{j=1,\ldots,d'}$

Subsystems



System AB Composite system

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$$

Tensor product

basis

$$\{|a_i\rangle_A \otimes |b_j\rangle_B\}_{j=1,\ldots,d'}^{i=1,\ldots,d}$$

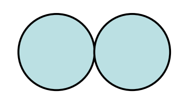
$$\dim(\mathcal{H}_A \otimes \mathcal{H}_B) = \dim \mathcal{H}_A \dim \mathcal{H}_B$$

Basic rule IV: Compositions

When system A and system B are independently accessed …

	State preparation	Unitary transformation	Orthogonal measurement
System A	$\|\phi\rangle_A$	$\hat{U}_A$	$\{\|a_i\rangle_A\}_{i=1,\cdots,d_A}$
System B	$\|\psi\rangle_B$	$\hat{V}_B$	$\{\|b_j\rangle_B\}_{j=1,\cdots,d_B}$
System AB	$\|\phi\rangle_A \otimes \|\psi\rangle_B$	$\hat{U}_A \otimes \hat{V}_B$	$\{\|a_i\rangle_A \otimes \|b_j\rangle_B\}_{i=1,\cdots,d_A}^{j=1,\cdots,d_B}$
	Separable states	Local unitary operations	Local measurements

When system A and system B are directly interacted …



$\|\Psi\rangle_{AB} \in \mathcal{H}_{AB}$ $\sum_k \alpha_k \|\phi_k\rangle_A \otimes \|\psi_k\rangle_B$	$\hat{U}_{AB} : \mathcal{H}_{AB} \to \mathcal{H}_{AB}$	$\{\|\Psi_k\rangle_{AB}\}_{k=1,2,\ldots,d_A d_B}$
Entangled states	Global unitary operations	Global measurements

Basic rule V: Causality

The marginal state of a subsystem is not changed by operating on other subsystems, as long as no information on the outcome of the operation is referred to.

The marginal state does not change.

The marginal state = the state that the subsystem would be in if we discard all the other constituent subsystems.

2. Density operators

Measurement on a subsystem

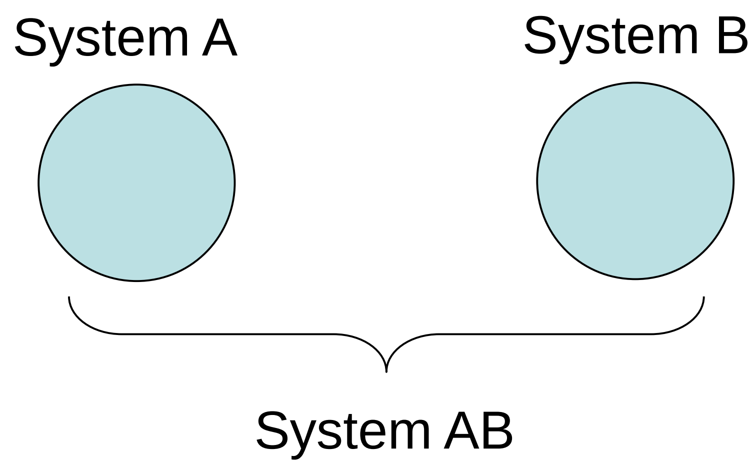
Marginal state of a subsystem

Density operators

Properties of bipartite pure states

Physical states and density operotors

Entanglement



state: $|\Phi\rangle_{AB}$

Suppose that the whole system (AB) is in a pure state.
We know everything that we can about the system AB.

Intuition in a ‘classical’ world:

If the whole is well known, so are the parts.

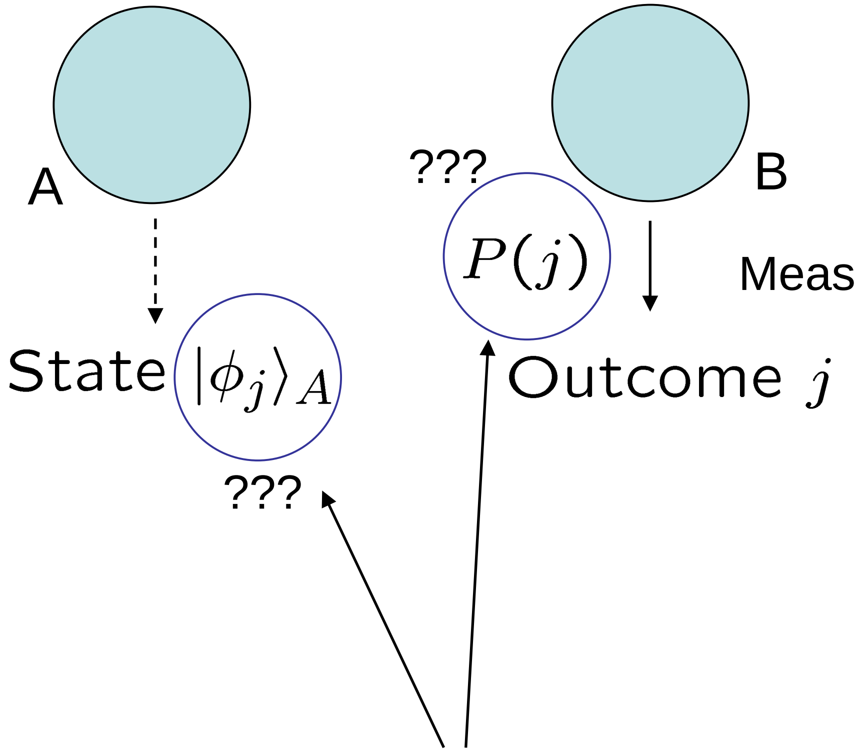
But ….

When system AB is entangled, the state of subsystem A is not a pure state.

What is the state of subsystem A?

Measurement on a subsystem

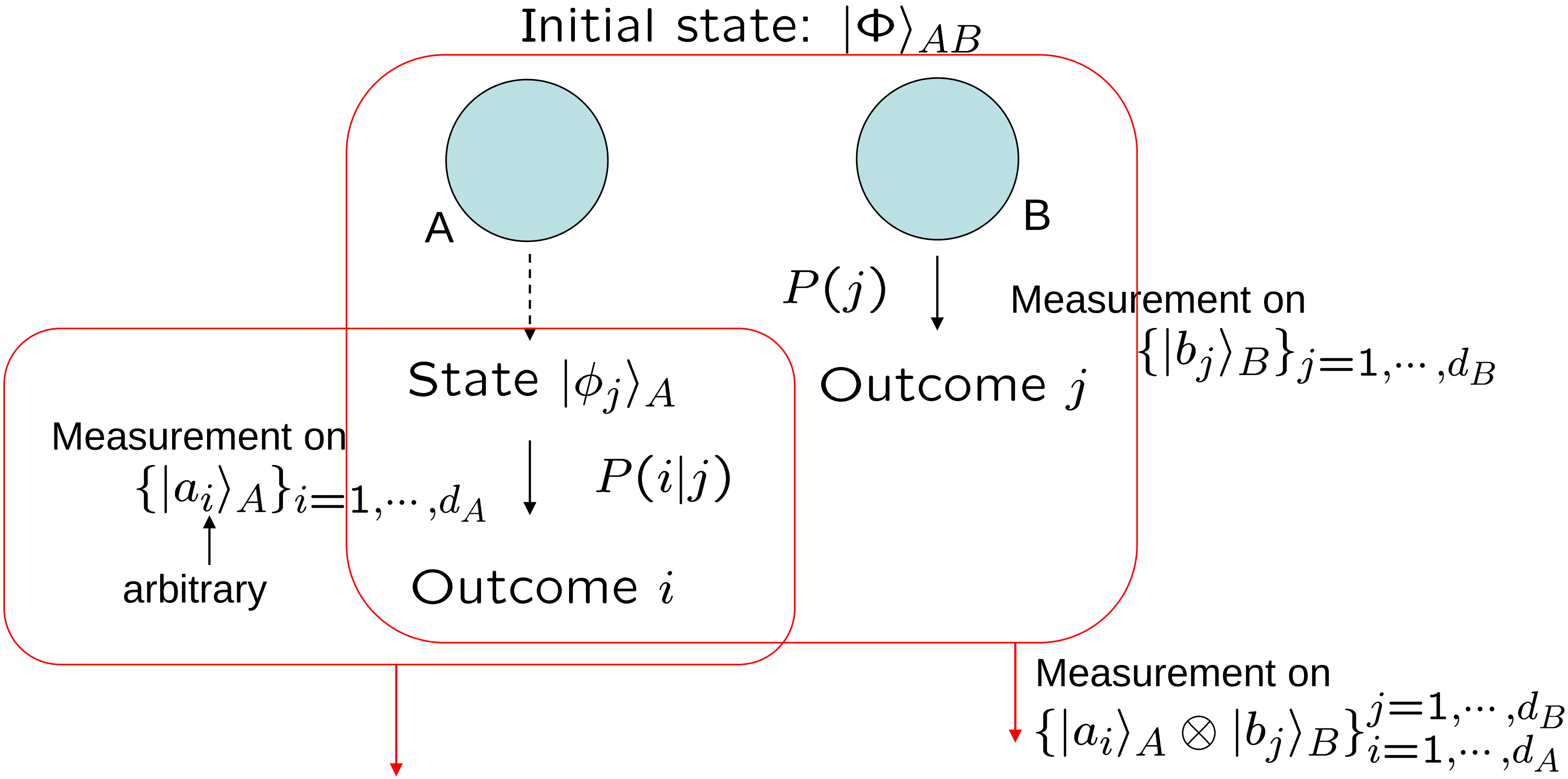
Initial state: $|\Phi\rangle_{AB}$



Measurement on
$\{|b_j\rangle_B\}_{j=1,\cdots,d_B}$

What is the rule to determine these?
Let us derive it from the basic rules.

Measurement on a subsystem



$$P(i|j) = |{}_A\langle a_i|\phi_j\rangle_A|^2$$

$$P(i,j) = \left|{}_A\langle a_i|{}_B\langle b_j||\Phi\rangle_{AB}\right|^2$$

$$P(i,j) = P(i|j)P(j) = |{}_A\langle a_i|\sqrt{P(j)}|\phi_j\rangle_A|^2$$

A remark on notations

$$ {}_A\langle a_i| \otimes {}_B\langle b_j||\Phi\rangle_{AB} $$

$$ = {}_A\langle a_i|\underbrace{(\hat{1}_A \otimes {}_B\langle b_j|)}_{\downarrow\ \text{abbreviation}}|\Phi\rangle_{AB} $$

$$ = {}_A\langle a_i|\ {}_B\langle b_j|\ |\Phi\rangle_{AB} $$

$$ \begin{array}{l} {}_A\langle a_i| \\ {}_B\langle b_j| \end{array} \Big|\Phi\Big\rangle_{AB} $$

$$ \begin{array}{ll} {}_A\langle a_i| & \\ & {}_B\langle b_j| \end{array} \Big|\Phi\Big\rangle_{AB} $$

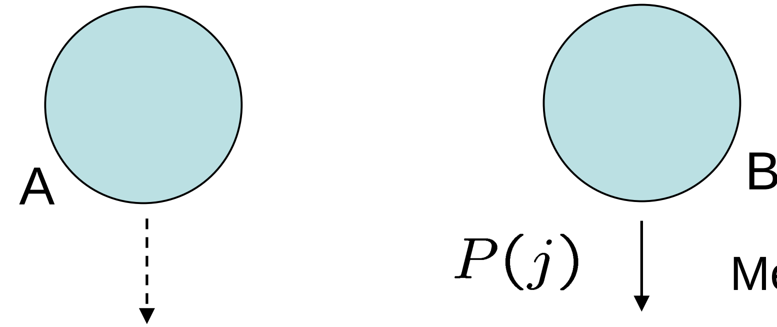
$$ {}_B\langle b_j| : \mathcal{H}_B \to \mathbb{C} $$

$$ \hat{1}_A : \mathcal{H}_A \to \mathcal{H}_A $$

$$ \hat{1}_A \otimes {}_B\langle b_j| : \mathcal{H}_A \otimes \mathcal{H}_B \to \mathcal{H}_A $$

Measurement on a subsystem

Initial state: $|\Phi\rangle_{AB}$



Measurement on $\{|b_j\rangle_B\}_{j=1,\cdots,d_B}$

State $|\phi_j\rangle_A$ Outcome j

For arbitrary $\{|a_i\rangle_A\}_{i=1,\cdots,d_A}$

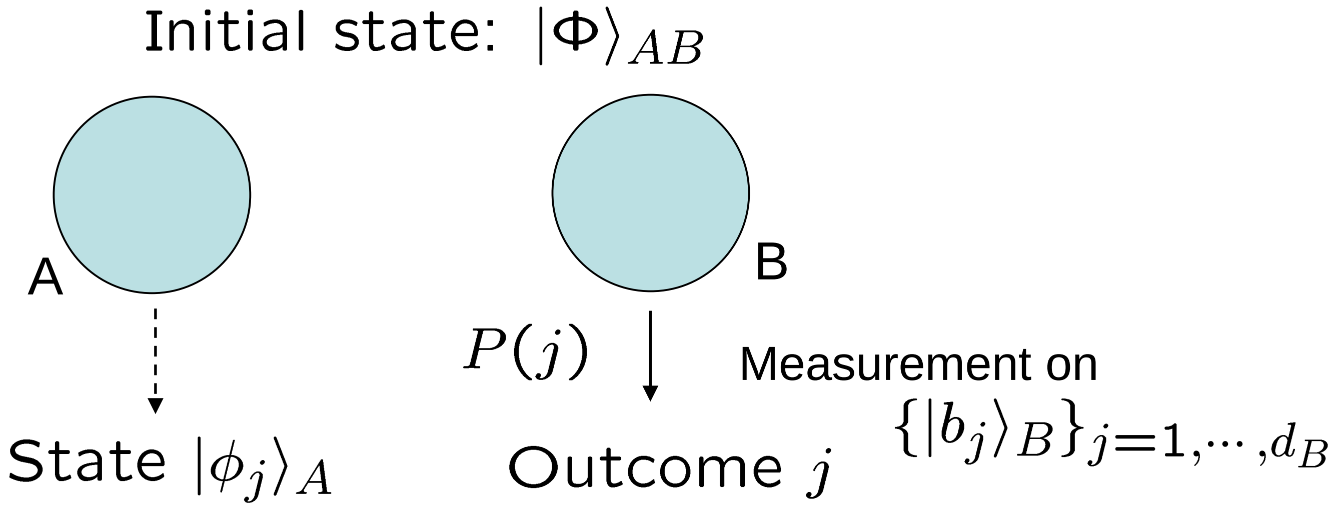
$$P(i,j) = \left|{}_A\langle a_i|_B\langle b_j||\Phi\rangle_{AB}\right|^2$$

$$P(i,j) = P(i|j)P(j) = |{}_A\langle a_i|\sqrt{P(j)}|\phi_j\rangle_A|^2$$

$$\downarrow$$

$$\sqrt{P(j)}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

Rule: Measurement on a subsystem



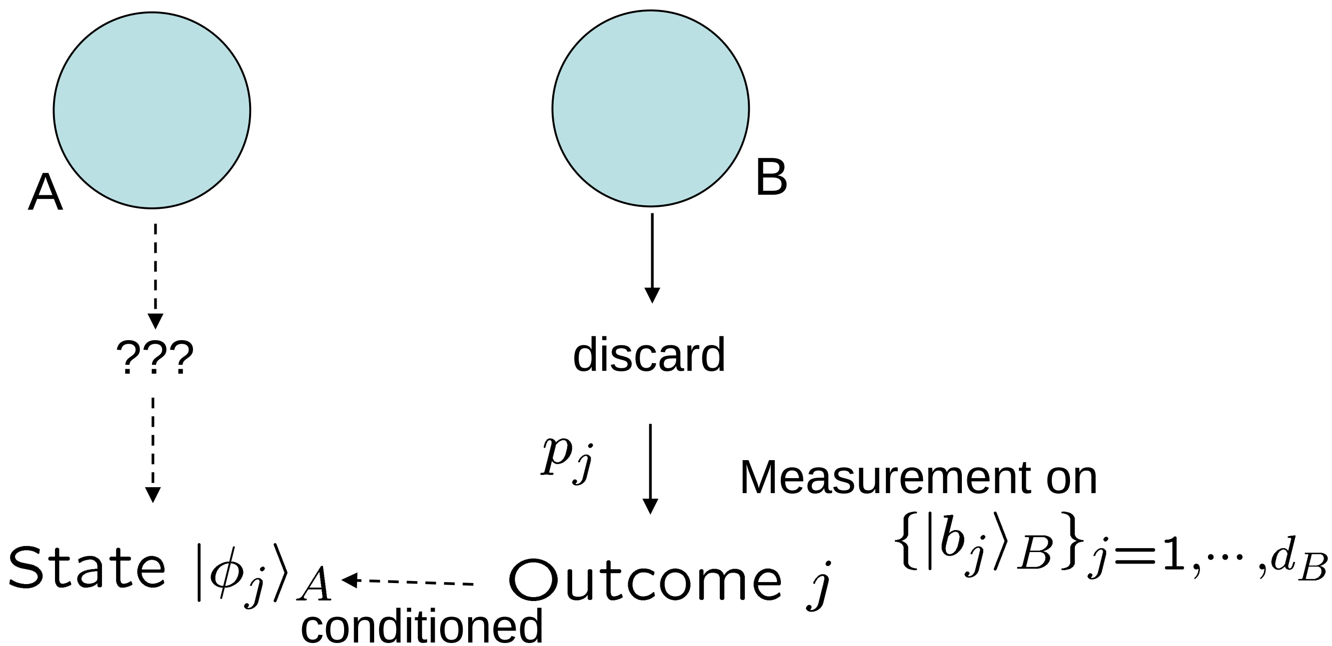
The measurement on system B produces outcome j with probability $P(j)$, and, conditioned on j, the subsystem A behaves as if it was initially prepared in the pure state $|\phi_j\rangle$, where

$$\boxed{\sqrt{P(j)}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}}$$

$$P(j) = \|{}_B\langle b_j||\Phi\rangle_{AB}\|^2 \qquad |\phi_j\rangle_A = \frac{{}_B\langle b_j||\Phi\rangle_{AB}}{\|{}_B\langle b_j||\Phi\rangle_{AB}\|}$$

Marginal state of a subsystem
(State after discarding the other subsystems)

Initial state: $|\Phi\rangle_{AB}$



State of system A: $|\phi_j\rangle_A$ with probability p_j $\longrightarrow$ $\{p_j, |\phi_j\rangle_A\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

This description is correct, but dependence on the fictitious measurement is weird…

Alternative description: density operator

$\{p_j, |\phi_j\rangle_A\}$ $\qquad$ $|\phi_j\rangle_A$ with probability p_j

$$\hat{\rho}_A \equiv \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j|$$

Cons

$\{q_k, |\psi_k\rangle_A\}$ $\rightarrow$ Same $\hat{\rho}_A$

$\{p_j, |\phi_j\rangle_A\}$ $\rightarrow$

Two different physical states could have the same density operator.
(The description could be insufficient.)

Pros

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

$$\hat{\rho}_A = \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j| = \sum_j \sqrt{p_j}|\phi_j\rangle_{AA}\langle\phi_j|\sqrt{p_j}$$

$$= \sum_j {}_B\langle b_j||\Phi\rangle\langle\Phi||b_j\rangle_B = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Independent of the choice of the fictitious measurement

Properties of density operators

$$\hat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

For any $|\psi\rangle$, $\langle\psi|\hat{\rho}|\psi\rangle = \sum_j p_j |\langle\psi|\phi_j\rangle|^2 \geq 0$ Positive

$$\mathrm{Tr}(\hat{\rho}) = \sum_j p_j \mathrm{Tr}(|\phi_j\rangle\langle\phi_j|)$$
$$= \sum_j p_j \langle\phi_j|\phi_j\rangle = \sum_j p_j = 1$$ Unit trace

Positive & Unit trace $\longrightarrow$ $\hat{\rho} = \sum_j \underset{\text{probability}}{\underset{\uparrow}{p_j}} |\phi_j\rangle\langle\phi_j|$

This decomposition is by no means unique!

Mixed state $\hat{\rho} = \sum_j p_j |\phi_j\rangle\langle\phi_j|$

Maximally mixed state $\hat{\rho} = d^{-1}\hat{1}$ (All eigenvalue are d^{-1})

$d = \dim \mathcal{H}$

Pure state $\hat{\rho} = |\phi\rangle\langle\phi|$ (One eigenvalue is 1)

Rules in terms of density operators

Prepare $|\phi_j\rangle$ with probability p_j

$$\hat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

Prepare $\hat{\rho}_j$ with probability p_j

$$\hat{\rho} = \sum_j p_j \hat{\rho}_j$$

Unitary evolution

$$|\phi_{\text{out}}\rangle = \hat{U}|\phi_{\text{in}}\rangle$$

Hint: $|\phi_{\text{out}}\rangle\langle\phi_{\text{out}}| = \hat{U}|\phi_{\text{in}}\rangle\langle\phi_{\text{in}}|\hat{U}^\dagger$

$$\hat{\rho}_{\text{out}} = \hat{U}\hat{\rho}_{\text{in}}\hat{U}^\dagger$$

Orthogonal measurement on basis $\{|a_j\rangle\}$

$$P(j) = |\langle a_j|\phi\rangle|^2$$

Hint: $P(j) = \langle a_j|\phi\rangle\langle\phi|a_j\rangle$

$$P(j) = \langle a_j|\hat{\rho}|a_j\rangle$$

Expectation value of an observable $\hat{A}$

$$\langle\hat{A}\rangle = \langle\phi|\hat{A}|\phi\rangle$$

$$\langle\hat{A}\rangle = \text{Tr}(\hat{A}\hat{\rho})$$

Hint: $\langle\hat{A}\rangle = \text{Tr}(\hat{A}|\phi\rangle\langle\phi|)$

Rules in terms of density operators

Independently prepared systems A and B

$$|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B \qquad\qquad \widehat{\rho}_{AB} = \widehat{\rho}_A \otimes \widehat{\rho}_B$$

Local measurement on system B on basis $\{|b_j\rangle_B\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB} \qquad\qquad p_j\widehat{\rho}_A^{(j)} = {}_B\langle b_j|\widehat{\rho}_{AB}|b_j\rangle_B$$

Discarding system B

$$\widehat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|) \qquad\qquad \widehat{\rho}_A = \mathrm{Tr}_B[\widehat{\rho}_{AB}]$$

All the rules so far can be written in terms of density operators.

Which is the better description?

$$\{p_j, |\phi_j\rangle\}$$

This looks natural. The system is in one of the pure states, but we just don't know. Quantum mechanics may treat just the pure states, and leave mixed states to statistical mechanics or probability theory.

$$\widehat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

Best description

All the rules so far can be written in terms of density operators.

Which description has one-to-one correspondence to physical states?

Theorem: Two states $\{p_j, |\phi_j\rangle\}$ and $\{q_k, |\psi_k\rangle\}$ with the same density operator are physically indistinguishable (hence are the same state).

Schmidt decomposition

Bipartite pure states have a very nice standard form.

Any orthonormal bases $\{|a_i\rangle_A\}$ $\{|b_j\rangle_B\}$

$$|\Phi\rangle_{AB} = \sum_{ij} \alpha_{ij} |a_i\rangle_A |b_j\rangle_B$$

We can always choose the two bases such that

$$|\Phi\rangle_{AB} = \sum \sqrt{p_i} |a_i\rangle_A |b_i\rangle_B$$ Schmidt decomposition

Example $|\Psi\rangle_{AB} = \sqrt{p}|a_0\rangle_A|b_0\rangle_B + \sqrt{1-p}|a_1\rangle_A|b_1\rangle_B$

$\{|a_i\rangle_A\}$: Diagonalizes $\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$

Proof: $|\Phi\rangle_{AB} = \sum_i |a_i\rangle_A |\tilde{b}_i\rangle_B$ $\quad$ $|\tilde{b}_i\rangle_B \equiv {}_A\langle a_i||\Phi\rangle_{AB}$

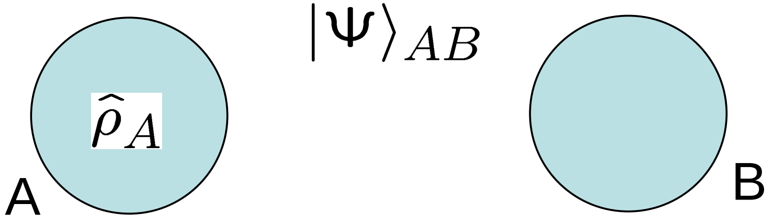
unnormalized

$$\begin{aligned} {}_B\langle \tilde{b}_j|\tilde{b}_i\rangle_B &= \mathrm{Tr}[{}_A\langle a_i||\Phi\rangle_{AB\,AB}\langle\Phi||a_j\rangle_A] \\ &= {}_A\langle a_i|\mathrm{Tr}_B[|\Phi\rangle_{AB\,AB}\langle\Phi|]|a_j\rangle_A \\ &= {}_A\langle a_i|\hat{\rho}_A|a_j\rangle_A = p_j\delta_{ij}. \end{aligned}$$

$\sqrt{p_j}|b_j\rangle \equiv |\tilde{b}_j\rangle_B$

Pure states with the same marginal state

$$|\Phi\rangle_{AB}$$
$$|\Psi\rangle_{AB}$$

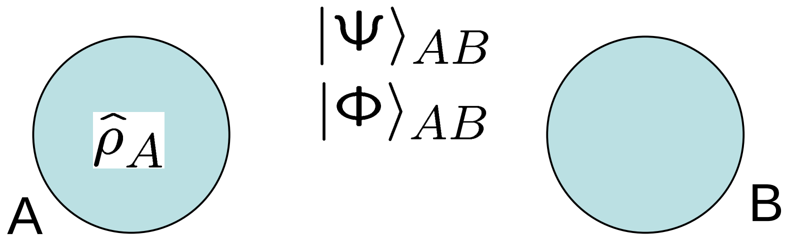


$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$
$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|)$$

$|\Phi\rangle_{AB} \longrightarrow \hat{\rho}_A$ Marginal state (unique)

$\hat{\rho}_A \longrightarrow |\Phi\rangle_{AB}$, $\hat{\rho}_A \longrightarrow |\Psi\rangle_{AB}$ Purification
Pure Extension (not unique)

Pure states with the same marginal state



$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Schmidt decomposition

Orthonormal basis $\{|a_i\rangle_A\}$ that diagonalizes $\hat{\rho}_A$

$$|\Psi\rangle_{AB} = \sum_i \sqrt{p_i}|a_i\rangle_A|\mu_i\rangle_B$$

$$|\Phi\rangle_{AB} = \sum_i \sqrt{p_i}|a_i\rangle_A|\nu_i\rangle_B$$

$\{|\mu_i\rangle_B\}$ Orthonormal basis

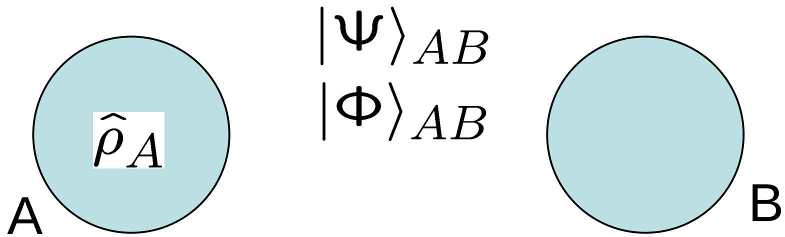
$\{|\nu_i\rangle_B\}$ Orthonormal basis

$$|\nu_i\rangle_B = \hat{U}_B|\mu_i\rangle_B$$

unitary

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

Pure states with the same marginal state



$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

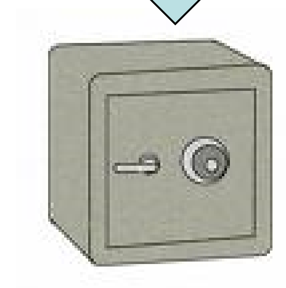
$$\boxed{|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}}$$

Theorem: If $|\Psi\rangle_{AB}$ and $|\Phi\rangle_{AB}$ are purifications of the same state $\hat{\rho}_A$, state $|\Psi\rangle_{AB}$ can be physically converted to state $|\Phi\rangle_{AB}$ without touching system A.

Sealed move (封じ手)

Chess, Go, Shogi …

Bb5
4六銀



Let us call it a day and shall we start over tomorrow, with Bob’s move.

While they are (suppose to be) sleeping...

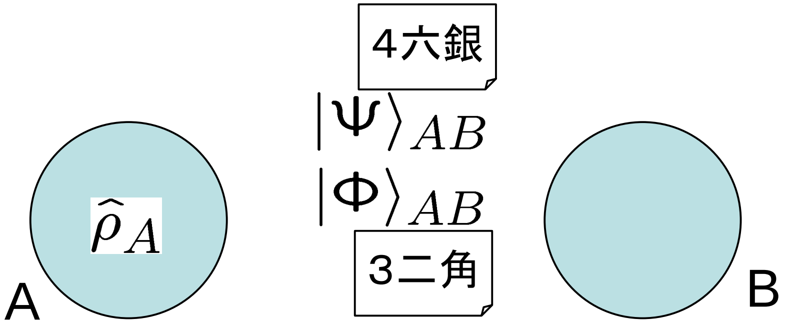
- Alice should not learn the sealed move.
- Bob should not alter the sealed move.

Sealed move

- Alice should not learn the sealed move.
- Bob should not alter the sealed move.

If there is no reliable safe available …

(If there is no system out of both Alice's and Bob's reach …)



Alice has no knowledge

↓

Bob can alter the states

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$\boxed{|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}}$$

Function of the "safe" cannot be realized.

Impossibility of unconditionally secure quantum bit commitment

(Lo, Mayers)

Mixed states with the same density operator

$\{p_j, |\phi_j\rangle_A\}$ $\quad$ $|\phi_j\rangle_A$ with probability p_j

$\{q_k, |\psi_k\rangle_A\}$ $\quad$ $|\psi_k\rangle_A$ with probability q_k

$$\hat{\rho}_A \equiv \sum_j p_j |\phi_j\rangle_{AA}\langle\phi_j| = \sum_k q_k |\psi_k\rangle_{AA}\langle\psi_k|$$

A scheme to prepare $\{p_j, |\phi_j\rangle_A\}$

Prepare system AB in state

$\{|b_j\rangle_B\}$ Orthonormal basis

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j}|\phi_j\rangle_A |b_j\rangle_B$$

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

Measure system B on basis $\{|b_j\rangle_B\}$

$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB}$$

$|\phi_j\rangle_A$ with probability p_j

Mixed states with the same density operator

A scheme to prepare $\{p_j, |\phi_j\rangle_A\}$

Prepare system AB in state

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j}|\phi_j\rangle_A|b_j\rangle_B$$

Measure system B on basis $\{|b_j\rangle_B\}$

$|\phi_j\rangle_A$ with probability p_j

$\{p_j, |\phi_j\rangle_A\}$

A scheme to prepare $\{q_k, |\psi_k\rangle_A\}$

Prepare system AB in state

$$|\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A|b_k\rangle_B$$

Measure system B on basis $\{|b_k\rangle_B\}$

$|\psi_k\rangle_A$ with probability q_k

$\{q_k, |\psi_k\rangle_A\}$

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

Mixed states with the same density operator

Prepare system AB in state

$$|\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A|b_k\rangle_B$$

Apply unitary operation $\hat{U}_B$ to system B

$$|\Phi\rangle_{AB} \equiv \sum_j \sqrt{p_j}|\phi_j\rangle_A|b_j\rangle_B$$

Measure system B on basis $\{|b_j\rangle_B\}$

$|\phi_j\rangle_A$ with probability p_j

$$\{p_j, |\phi_j\rangle_A\}$$

$$|\Psi\rangle_{AB} \equiv \sum_k \sqrt{q_k}|\psi_k\rangle_A|b_k\rangle_B$$

Measure system B on basis $\{|b_k\rangle_B\}$

$|\psi_k\rangle_A$ with probability q_k

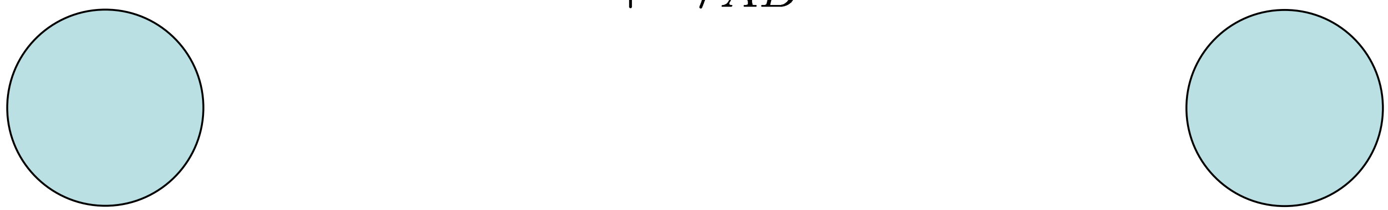
$$\{q_k, |\psi_k\rangle_A\}$$

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Psi\rangle\langle\Psi|) = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|)$$

$$|\Phi\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Psi\rangle_{AB}$$

Mixed states with the same density operator

$|\Psi\rangle_{AB}$



A $\{p_j, |\phi_j\rangle_A\}$

$\{q_k, |\psi_k\rangle_A\}$

B

Alice

Bob

The marginal state of system A should not be changed by operations on system B

Bob can remotely decide which of the states the system A is in.

Theorem: Two states $\{p_j, |\phi_j\rangle\}$ and $\{q_k, |\psi_k\rangle\}$ with the same density operator are physically indistinguishable (hence are the same state).

Density operator

$\updownarrow$ One-to-one

Physical state

Example

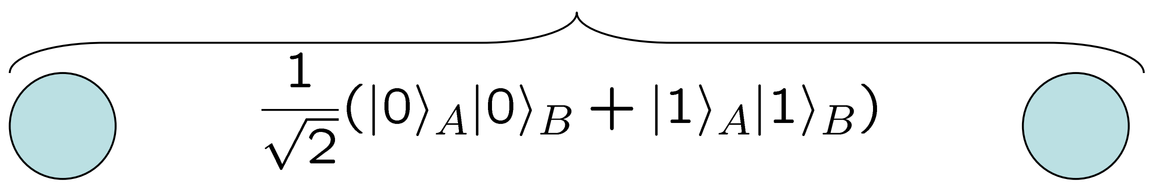
$\{|0\rangle_A, |1\rangle_A\}$: an orthonormal basis $\qquad |\pm\rangle_A \equiv \frac{1}{\sqrt{2}}(|0\rangle_A \pm |1\rangle_A)$

$\{|+\rangle_A, |-\rangle_A\}$: an orthonormal basis

Recipe I: $\{p_j, |\phi_j\rangle_A\}$ $\quad p_0 = p_1 = \frac{1}{2},\ |\phi_0\rangle_A = |0\rangle_A, |\phi_1\rangle_A = |1\rangle_A$

Recipe II: $\{q_k, |\psi_k\rangle_A\}$ $\quad q_0 = q_1 = \frac{1}{2},\ |\psi_0\rangle_A = |+\rangle_A, |\psi_1\rangle_A = |-\rangle_A$

$$\frac{1}{2}|0\rangle_{AA}\langle 0| + \frac{1}{2}|1\rangle_{AA}\langle 1| \ = \ \frac{1}{2}|+\rangle_{AA}\langle +| + \frac{1}{2}|-\rangle_{AA}\langle -| \ = \ \frac{1}{2}\hat{1}$$



$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

A $\qquad\qquad$ B

measurement

50% $\quad$ 50%

$\{|0\rangle_A, |1\rangle_A\}$ $\longleftarrow$ $\{|0\rangle_B, |1\rangle_B\}$

(Recipe I)

50% $\quad$ 50%

$\{|+\rangle_A, |-\rangle_A\}$ $\longleftarrow$ $\{|+\rangle_B, |-\rangle_B\}$

(Recipe II)

3. Generalized measurements and quantum operations

Use of auxiliary systems and Kraus representation

Generalized measurement and POVM

Quantum operation and CPTP map

Relation between quantum operations and bipartite states

What can we do in principle?

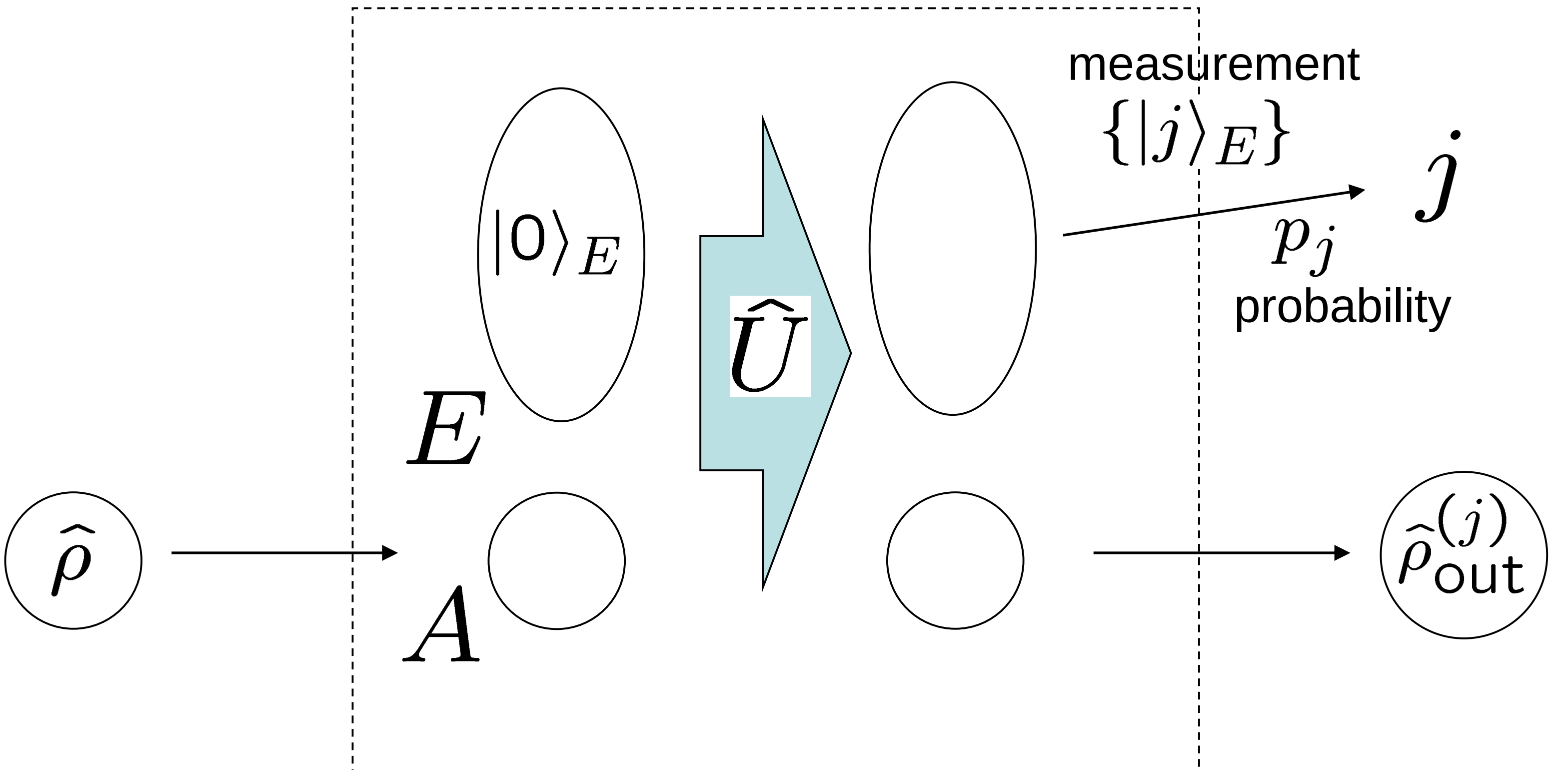
Measure of distinguishability

Use of auxiliary systems

Basic operations
- Unitary operations
- Orthogonal measurements

\+

An auxiliary system (ancilla)

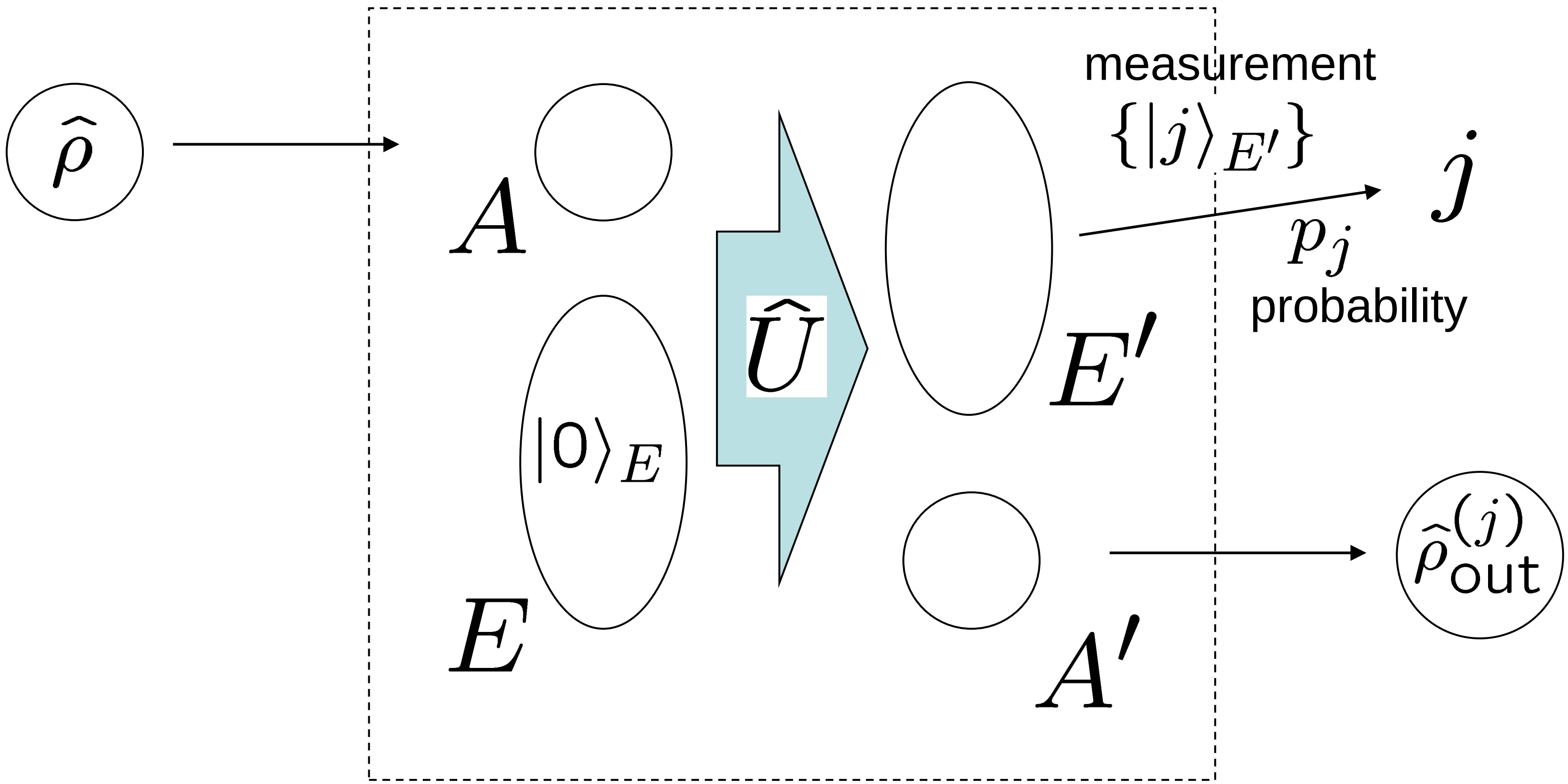


Use of auxiliary systems

Basic operations
Unitary operations
Orthogonal measurements

+

An auxiliary system
(ancilla)



Rules in terms of density operators

Prepare $|\phi_j\rangle$ with probability p_j

$$\hat{\rho} \equiv \sum_j p_j |\phi_j\rangle\langle\phi_j|$$

Prepare $\hat{\rho}_j$ with probability p_j

$$\hat{\rho} = \sum_j p_j \hat{\rho}_j$$

Unitary evolution

$$|\phi_{\text{out}}\rangle = \hat{U}|\phi_{\text{in}}\rangle$$

Hint: $|\phi_{\text{out}}\rangle\langle\phi_{\text{out}}| = \hat{U}|\phi_{\text{in}}\rangle\langle\phi_{\text{in}}|\hat{U}^\dagger$

$$\hat{\rho}_{\text{out}} = \hat{U}\hat{\rho}_{\text{in}}\hat{U}^\dagger$$

Orthogonal measurement on basis $\{|a_j\rangle\}$

$$P(j) = |\langle a_j|\phi\rangle|^2$$

Hint: $P(j) = \langle a_j|\phi\rangle\langle\phi|a_j\rangle$

$$P(j) = \langle a_j|\hat{\rho}|a_j\rangle$$

Expectation value of an observable $\hat{A}$

$$\langle\hat{A}\rangle = \langle\phi|\hat{A}|\phi\rangle$$

$$\langle\hat{A}\rangle = \text{Tr}(\hat{A}\hat{\rho})$$

Hint: $\langle\hat{A}\rangle = \text{Tr}(\hat{A}|\phi\rangle\langle\phi|)$

Rules in terms of density operators

Independently prepared systems A and B

$$|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\psi\rangle_B \qquad \hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$$

Local measurement on system B on basis $\{|b_j\rangle_B\}$

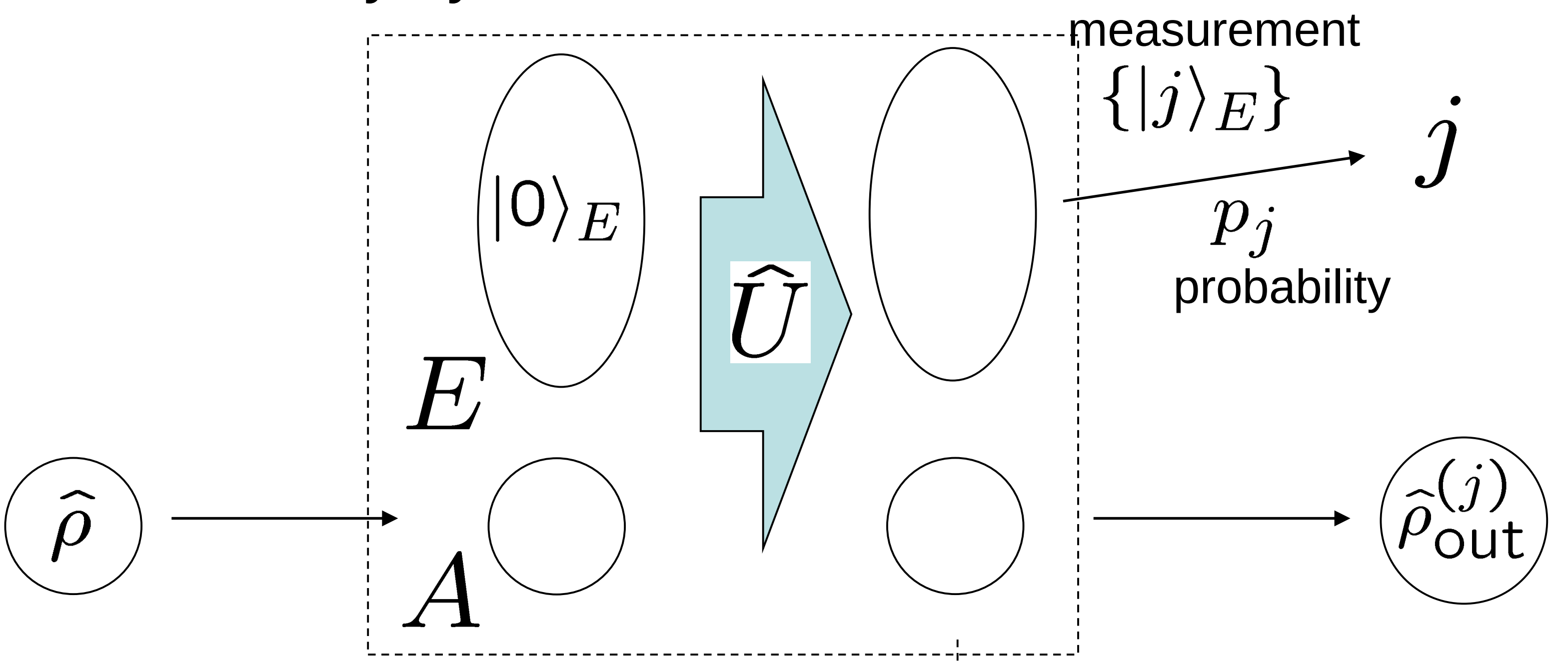
$$\sqrt{p_j}|\phi_j\rangle_A = {}_B\langle b_j||\Phi\rangle_{AB} \qquad p_j\hat{\rho}_A^{(j)} = {}_B\langle b_j|\hat{\rho}_{AB}|b_j\rangle_B$$

Discarding system B

$$\hat{\rho}_A = \mathrm{Tr}_B(|\Phi\rangle\langle\Phi|) \qquad \hat{\rho}_A = \mathrm{Tr}_B[\hat{\rho}_{AB}]$$

All the rules so far can be written in terms of density operators.

Use of auxiliary systems



$$\widehat{\rho} \otimes |0\rangle_{EE}\langle 0|$$

$$\widehat{U}(\widehat{\rho} \otimes |0\rangle_{EE}\langle 0|)\widehat{U}^\dagger$$

$$p_j \widehat{\rho}_{\text{out}}^{(j)} = {}_E\langle j|\widehat{U}(\widehat{\rho} \otimes |0\rangle_{EE}\langle 0|)\widehat{U}^\dagger|j\rangle_E$$

$$= \widehat{M}^{(j)} \widehat{\rho} \widehat{M}^{(j)\dagger}$$

$$\widehat{M}^{(j)} \equiv {}_E\langle j|\widehat{U}|0\rangle_E$$

$${}_E\langle j| \; \widehat{U} \; |0\rangle_E$$

$$\widehat{M}^{(j)} : \mathcal{H}_A \to \mathcal{H}_A$$

Kraus representation

$$p_j \hat{\rho}_{\text{out}}^{(j)} = {}_E\langle j|\hat{U}(\hat{\rho} \otimes |0\rangle_{EE}\langle 0|)\hat{U}^\dagger|j\rangle_E$$

$$\downarrow \quad \hat{M}^{(j)} \equiv {}_E\langle j|\hat{U}|0\rangle_E$$ Kraus operators

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)}\hat{\rho}\hat{M}^{(j)\dagger} \text{ with } \textstyle\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$$

Representation with no reference to the auxiliary system

$$\begin{aligned}
\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} &= \sum_j {}_E\langle 0|\hat{U}^\dagger|j\rangle_{EE}\langle j|\hat{U}|0\rangle_E \\
&= {}_E\langle 0|\hat{U}^\dagger\hat{U}|0\rangle_E \\
&= {}_E\langle 0|\hat{1}_A \otimes \hat{1}_E|0\rangle_E \\
&= \hat{1}_A
\end{aligned}$$

Kraus operators $\longrightarrow$ Physical realization

$$p_j \hat{\rho}^{(j)}_{\text{out}} = {}_E\langle j|\hat{U}(\hat{\rho} \otimes |0\rangle_{EE}\langle 0|)\hat{U}^\dagger|j\rangle_E$$

$$\hat{M}^{(j)} \equiv {}_E\langle j|\hat{U}|0\rangle_E$$ Kraus operators

$$p_j \hat{\rho}^{(j)}_{\text{out}} = \hat{M}^{(j)}\hat{\rho}\hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$$

Arbitrary set $\{\hat{M}^{(j)}\}$ satisfying $\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}$

$|\phi\rangle_A \otimes |0\rangle_E \mapsto \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$ is linear.

preserves inner products.

For any two states $|\phi\rangle_A$ and $|\psi\rangle_A$,

$$\left(\sum_{j'} \hat{M}^{(j')}|\psi\rangle_A \otimes |j'\rangle_E\right)^\dagger \left(\sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E\right)$$

$$= {}_A\langle\psi|\phi\rangle_A = (|\psi\rangle_A \otimes |0\rangle_E)^\dagger (|\phi\rangle_A \otimes |0\rangle_E) .$$

There exists a unitary satisfying

$\hat{U}(|\phi\rangle_A \otimes |0\rangle_E) = \sum_j \hat{M}^{(j)}|\phi\rangle_A \otimes |j\rangle_E$

Generalized measurement

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \ \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

$$p_j = \text{Tr}[\hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger}] = \text{Tr}[\hat{F}^{(j)} \hat{\rho}]$$

$$\hat{F}^{(j)} \equiv \hat{M}^{(j)\dagger} \hat{M}^{(j)} \geq 0$$

positive

$$\boxed{p_j = \text{Tr}[\hat{F}^{(j)} \hat{\rho}] \ \text{ with } \sum_j \hat{F}^{(j)} = \hat{1}}$$

$\{\hat{F}^{(j)}\}$ POVM

Positive operator valued measure

Generalized measurement

$$p_j = \mathrm{Tr}[\widehat{F}^{(j)}\widehat{\rho}] \text{ with } \textstyle\sum_j \widehat{F}^{(j)} = \hat{1}$$

Examples

Orthogonal measurement on basis $\{|a_j\rangle\}$

$$\widehat{F}^{(j)} = |a_j\rangle\langle a_j|$$

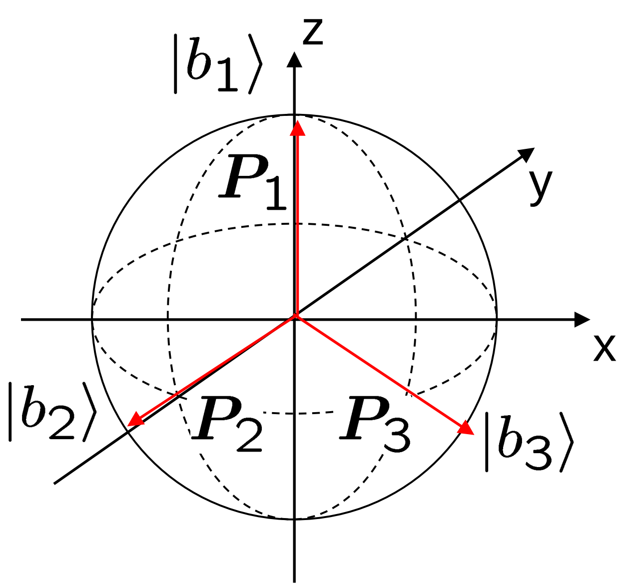
Trine measurement on a qubit

$$\widehat{F}^{(j)} = \tfrac{2}{3}|b_j\rangle\langle b_j|$$

$|b_j\rangle\langle b_j| = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$

$\sum_j \boldsymbol{P}_j = 0$

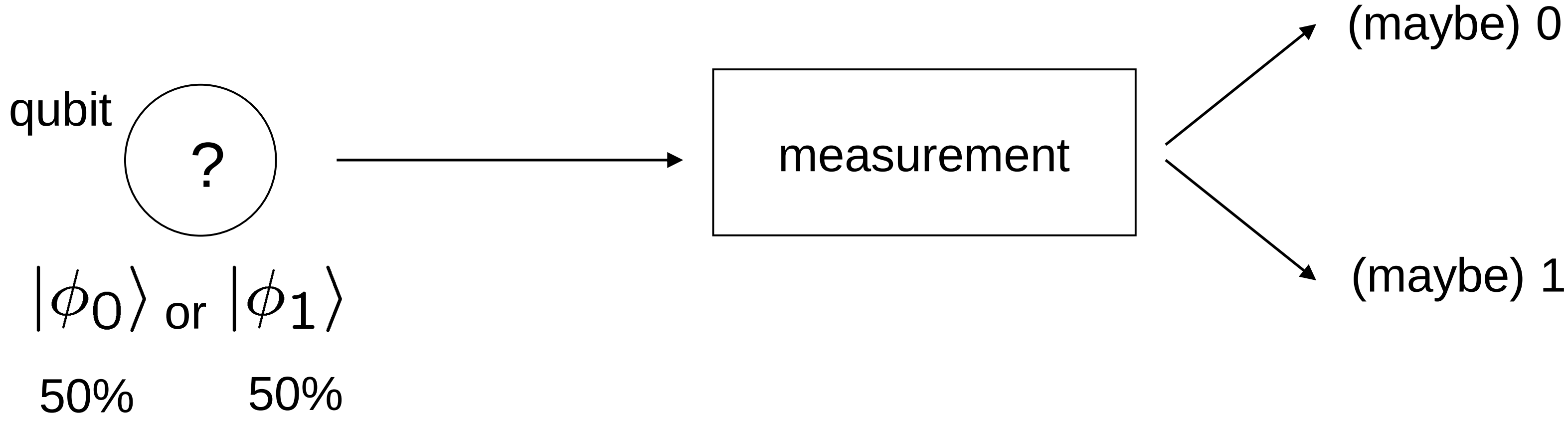
$$\longrightarrow \textstyle\sum_j \widehat{F}^{(j)} = \hat{1}$$



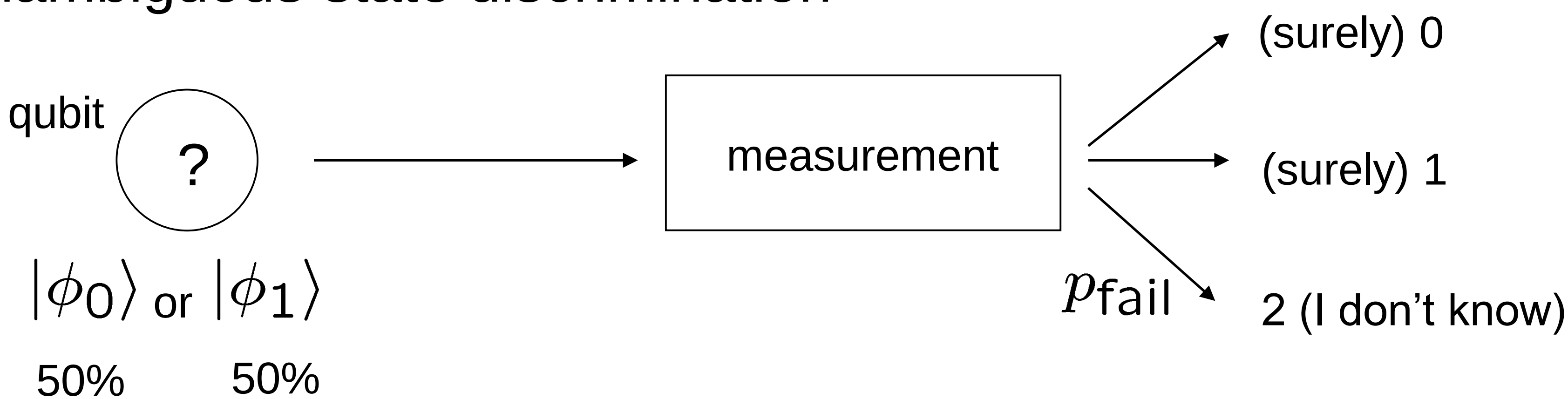
Distinguishing two nonorthogonal states

$$\langle \phi_0 | \phi_1 \rangle = s > 0$$

Minimum-error discrimination



Unambiguous state discrimination



Unambiguous state discrimination

qubit ? → measurement → (surely) 0; (surely) 1; p_{fail} 2 (I don't know)

$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%

$$\langle\phi_0|\phi_1\rangle = s > 0$$

Orthogonal measurement

$$\{|\phi_0\rangle, |\phi_0^{\perp}\rangle\}$$

2 (I don't know) (surely) 1

If the initial state is $|\phi_0\rangle$

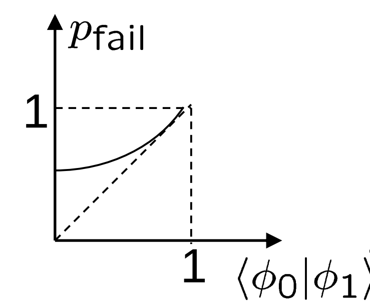
it always fails.

If the initial state is $|\phi_1\rangle$

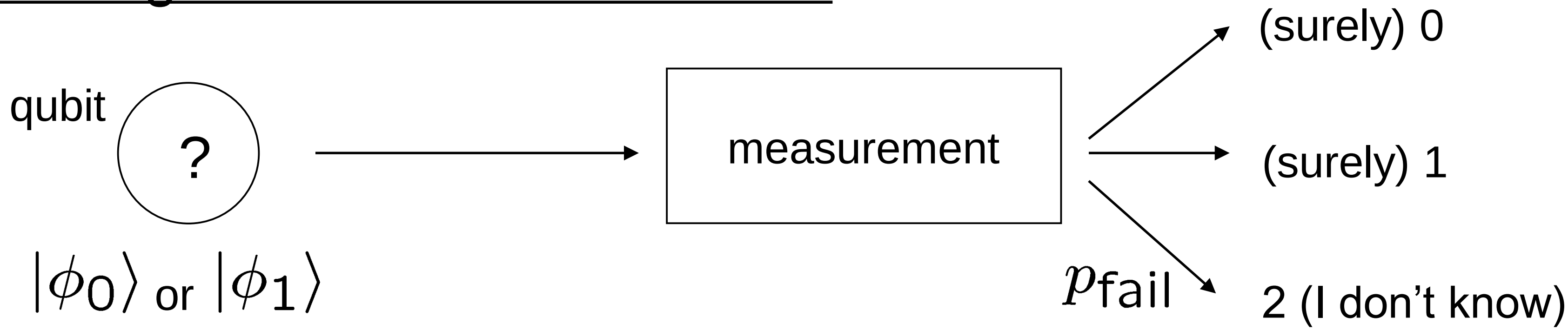
it fails with prob. $|\langle\phi_0|\phi_1\rangle|^2 = s^2$

$$\{|\phi_1\rangle, |\phi_1^{\perp}\rangle\}$$

$$p_{\text{fail}} = \frac{1+s^2}{2}$$



Unambiguous state discrimination



$|\phi_0\rangle$ or $|\phi_1\rangle$

50% 50%

$$\langle\phi_0|\phi_1\rangle = s > 0$$

Generalized measurement

$$\hat{F}_0 := \mu|\phi_1^\perp\rangle\langle\phi_1^\perp|$$

$$\hat{F}_1 := \mu|\phi_0^\perp\rangle\langle\phi_0^\perp|$$

$$\hat{F}_2 := \hat{1} - \hat{F}_0 - \hat{F}_1$$

The only constraint on μ comes from $\hat{F}_2 \geq 0$

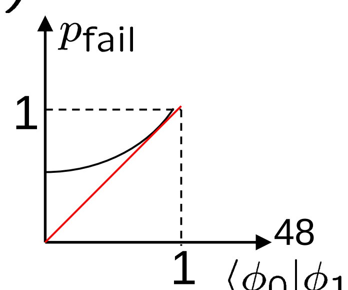
$$\langle\phi_0^\perp|\phi_1^\perp\rangle = s \qquad (\hat{F}_0 + \hat{F}_1 \leq \hat{1})$$

$$(\hat{F}_0 + \hat{F}_1)(|\phi_0^\perp\rangle \pm |\phi_1^\perp\rangle)$$
$$= \mu(1 \pm s)(|\phi_0^\perp\rangle \pm |\phi_1^\perp\rangle)$$

The optimum: $\mu = (1+s)^{-1}$

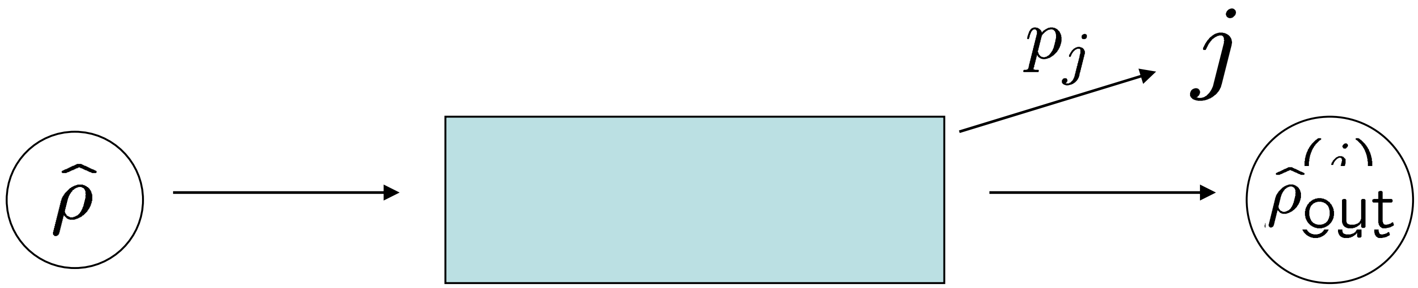
$$p_{\rm fail} = 1 - \frac{\mu}{2}|\langle\phi_0|\phi_1^\perp\rangle|^2 - \frac{\mu}{2}|\langle\phi_1|\phi_0^\perp\rangle|^2$$
$$= 1 - \mu(1 - s^2)$$

$$p_{\rm fail} = s$$



Quantum operation (Quantum channel, CPTP map)

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

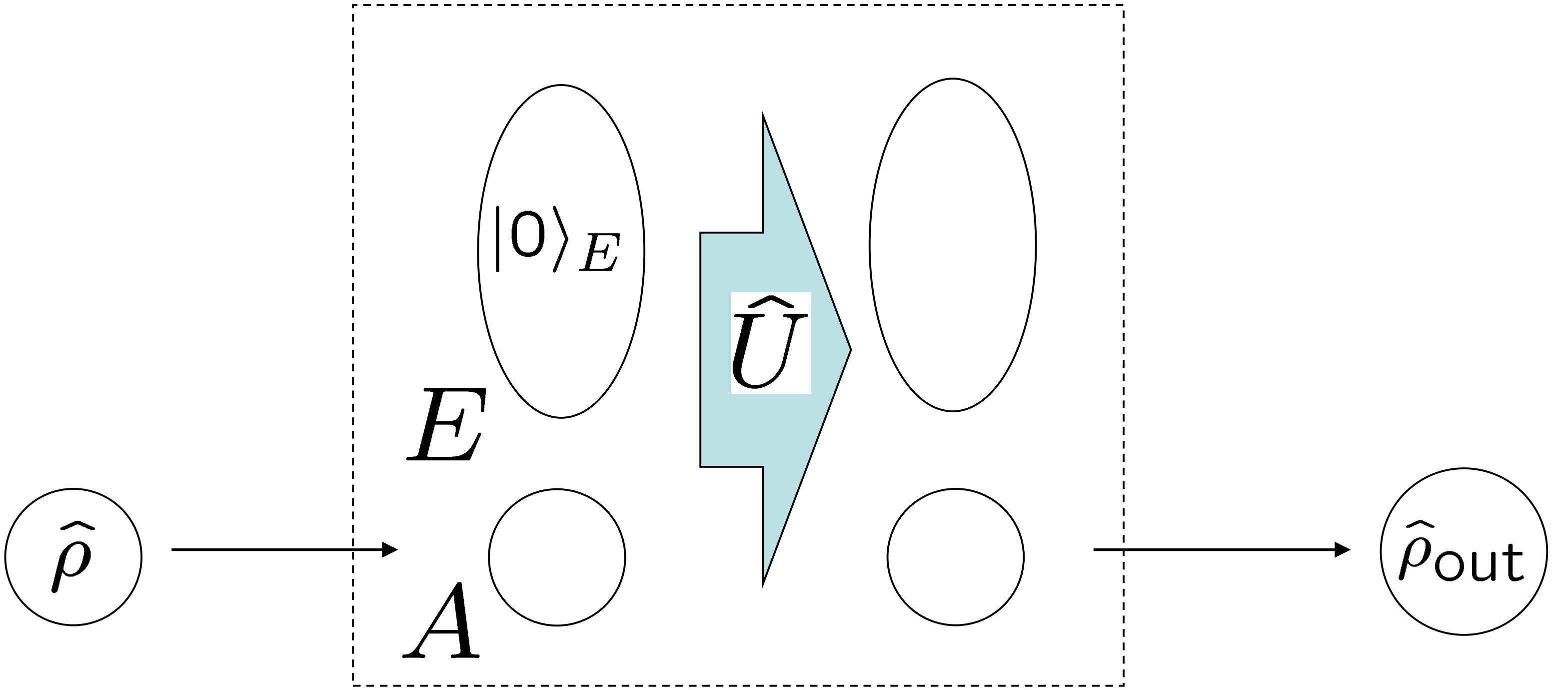


$$\hat{\rho}_{\text{out}} = \sum_j p_j \hat{\rho}_{\text{out}}^{(j)} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger}$$
$$= \sum_j {}_E\langle j | \hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger | j \rangle_E$$
$$= \text{Tr}_E[\hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger]$$

$$\boxed{\begin{aligned} \hat{\rho}_{\text{out}} &= \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \\ &= \text{Tr}_E[\hat{U} (\hat{\rho} \otimes |0\rangle_{EE}\langle 0|) \hat{U}^\dagger] \end{aligned}}$$

$\hat{\rho}_{\text{out}} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map
CPTP map

Quantum operation (Quantum channel, CPTP map)



$$\hat{\rho}_{\rm out} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \ \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$
$$= {\rm Tr}_E[\hat{U}(\hat{\rho} \otimes |0\rangle_{EE}\langle 0|)\hat{U}^\dagger]$$

$\hat{\rho}_{\rm out} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map
CPTP map

Dephasing channel

$\alpha|0\rangle + \beta|1\rangle$

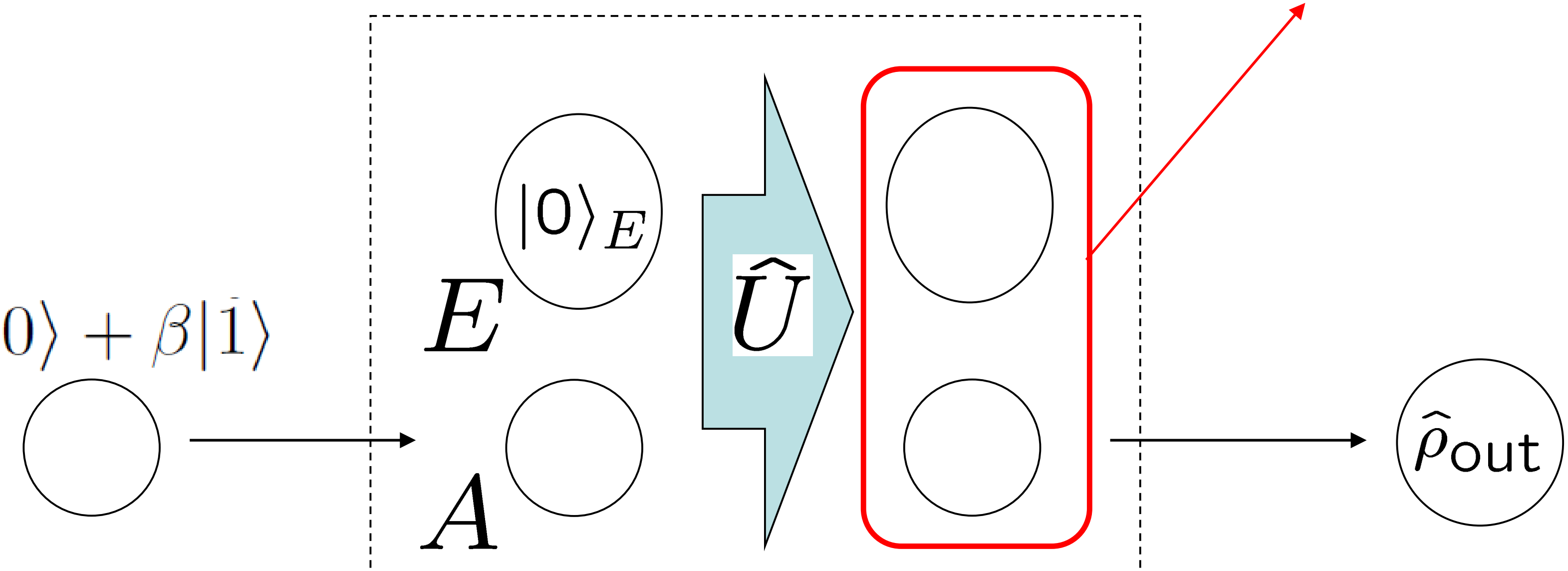
$\hat{\rho}_{\mathrm{out}} = |\alpha|^2|0\rangle\langle 0| + |\beta|^2|1\rangle\langle 1|$

$$\hat{M}^{(0)} = |0\rangle\langle 0|,\ \hat{M}^{(1)} = |1\rangle\langle 1|$$

$$\hat{M}^{(0)\dagger}\hat{M}^{(0)} + \hat{M}^{(1)\dagger}\hat{M}^{(1)} = \hat{1}$$

$\alpha|0\rangle \otimes |0\rangle_E + \beta|1\rangle \otimes |1\rangle_E$

$\alpha|0\rangle + \beta|1\rangle$



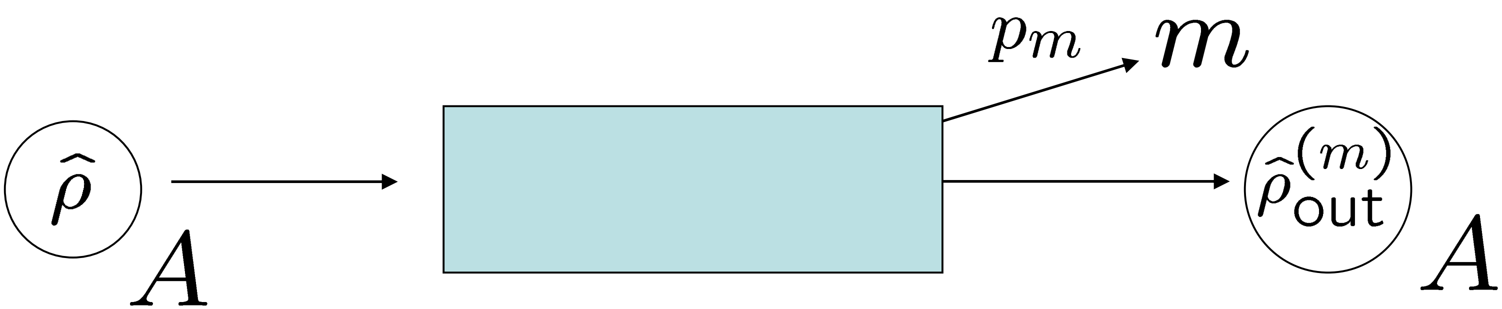
$$\hat{U}(|0\rangle \otimes |0\rangle_E) = |0\rangle \otimes |0\rangle_E$$
$$\hat{U}(|1\rangle \otimes |0\rangle_E) = |1\rangle \otimes |1\rangle_E$$

What can we do in principle?

We have seen what we can (at least) do by using an auxiliary system.

$$p_j \hat{\rho}_{\text{out}}^{(j)} = \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \text{ with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$

We also want to know what we cannot do.



Black box with classical and quantum outputs

Black box with quantum outputs

Maximally entangled states

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

Orthonormal bases $\{|k\rangle_A\}_{k=1,2,\ldots,d}$ $\{|k\rangle_B\}_{k=1,2,\ldots,d}$

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Maximally entangled state

Marginal states

$$\hat{\rho}_A = d^{-1}\hat{1}_A \qquad \hat{\rho}_B = d^{-1}\hat{1}_B$$

Maximally mixed state Maximally mixed state

Relative states

Fix a maximally entangled state

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

A $$|\Phi\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{k=1}^{d} |k\rangle_A |k\rangle_B$$ B

Relative states

$$|\phi\rangle_A = \sum_k \alpha_k |k\rangle_A \longleftrightarrow |\phi^*\rangle_B = \sum_k \overline{\alpha_k} |k\rangle_B$$

$$= \sqrt{d}\,{}_B\langle\phi^*||\Phi\rangle_{AB} \qquad\qquad = \sqrt{d}\,{}_A\langle\phi||\Phi\rangle_{AB}$$

$$|\Phi\rangle_{AB} \begin{cases} A \longrightarrow |\phi\rangle_A \\ B \xrightarrow{\text{Orthogonal measurement}} \text{outcome } j = 1 \end{cases}$$

$\{|v_j\rangle_B\}_{j=1,2,\ldots,d}$

$|v_1\rangle_B = |\phi^*\rangle_B$

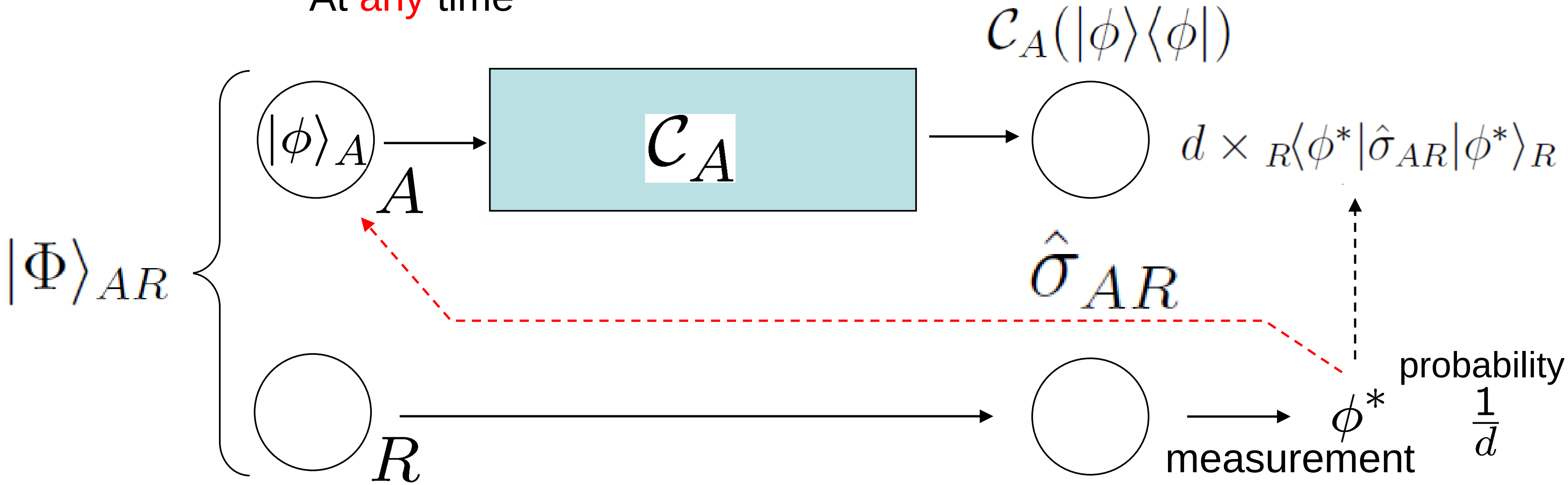
$$p_1 = \frac{1}{d}$$

$$\boxed{\frac{1}{\sqrt{d}} |\phi\rangle_A = {}_B\langle\phi^*||\Phi\rangle_{AB}}$$

Quantum operations and bipartite states

We can remotely prepare system A in any state with a nonzero success probability.

At any time



$$\hat{\sigma}_{AR} \equiv (\mathcal{C}_A \otimes \mathcal{I}_R)(|\Phi\rangle\langle\Phi|)$$

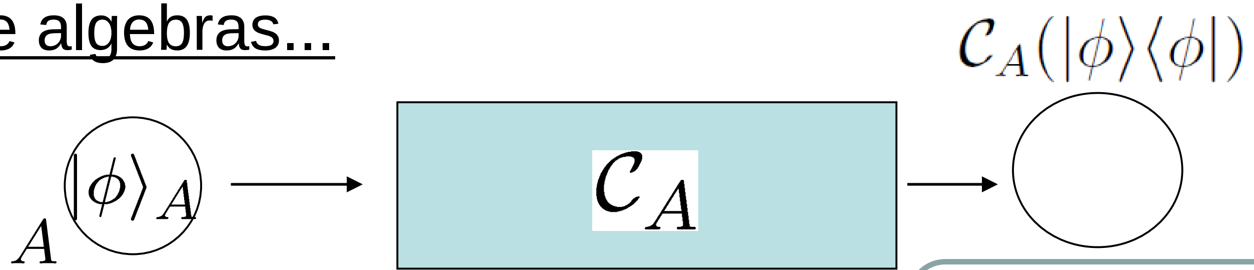
If this single state is known …

$$\mathcal{C}_A(|\phi\rangle\langle\phi|) = d \times {}_R\langle\phi^*|\hat{\sigma}_{AR}|\phi^*\rangle_R$$

Output for every input state is known!

Characterization of a process = Characterization of a state

Some algebras...



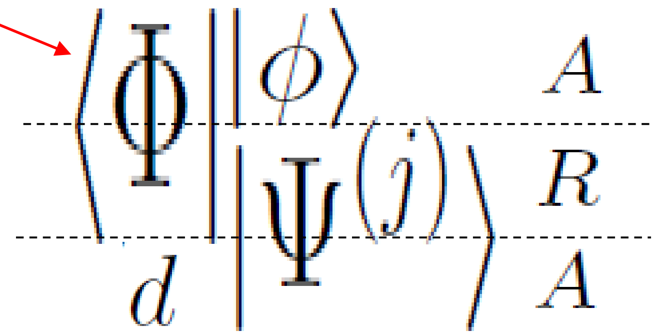
$$\mathcal{C}_A(|\phi\rangle\langle\phi|) = \sqrt{d}\ {}_R\langle\phi^*|\hat{\sigma}_{AR}|\phi^*\rangle_R \sqrt{d}$$

(unnormalized states)

$$\hat{\sigma}_{AR} = \sum_j |\Psi^{(j)}\rangle_{AR}\ {}_{AR}\langle\Psi^{(j)}|$$

$$= \sum_j \boxed{\sqrt{d}\ {}_R\langle\phi^*||\Psi^{(j)}\rangle_{AR}}\ {}_{AR}\langle\Psi^{(j)}||\phi^*\rangle_R \sqrt{d}$$

$$\langle\phi^*|_R = \sqrt{d}\ {}_{AR}\langle\Phi||\phi\rangle_A$$

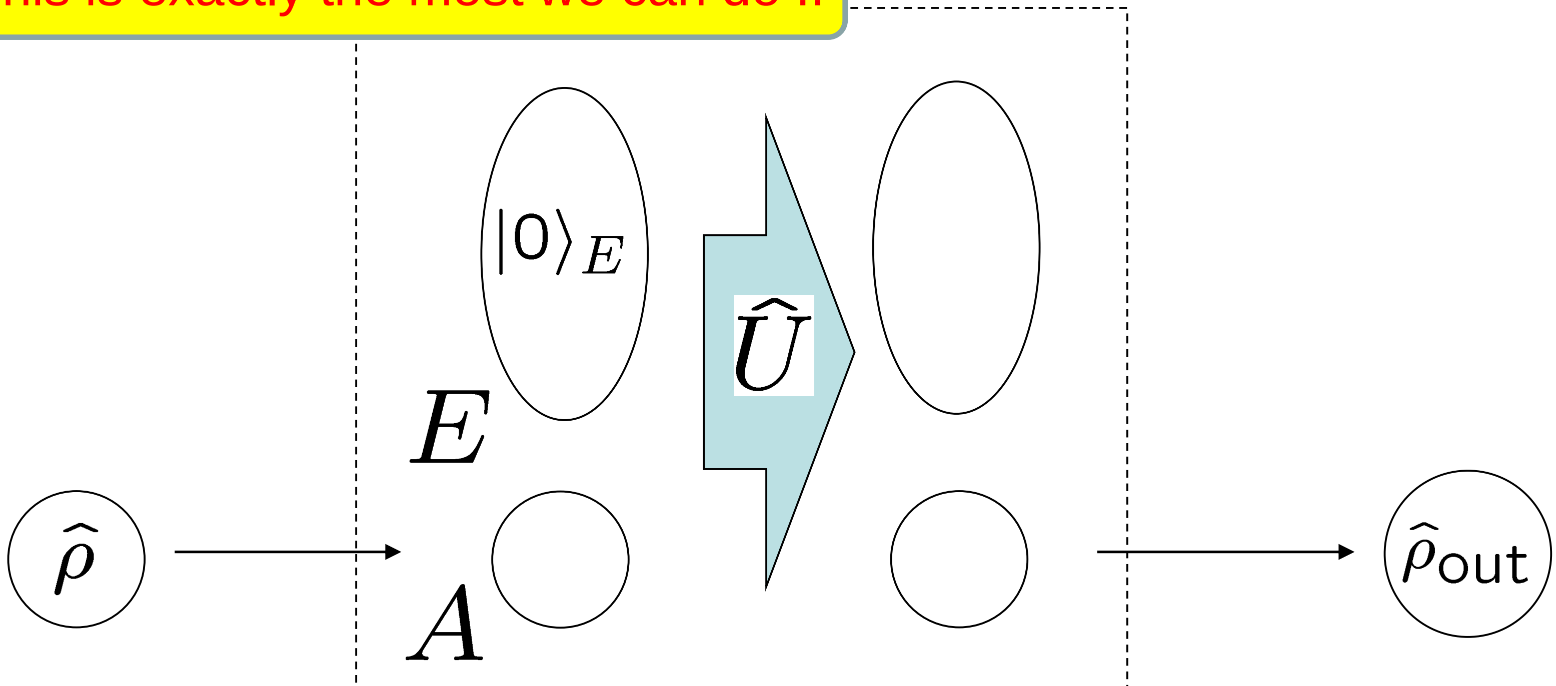


$$\boxed{\mathcal{C}_A(|\phi\rangle\langle\phi|) = \sum_j \hat{M}^{(j)}|\phi\rangle_A\ {}_A\langle\phi|\hat{M}^{(j)\dagger}}$$

Trace $\longrightarrow$ $1 = \sum_j {}_A\langle\phi|\hat{M}^{(j)\dagger}\hat{M}^{(j)}|\phi\rangle_A$ $\longrightarrow$ $\sum_j \hat{M}^{(j)\dagger}\hat{M}^{(j)} = \hat{1}_A$

Quantum operation (Quantum channel, CPTP map)

This is exactly the most we can do !!



$$\hat{\rho}_{\text{out}} = \sum_j \hat{M}^{(j)} \hat{\rho} \hat{M}^{(j)\dagger} \quad \text{with } \sum_j \hat{M}^{(j)\dagger} \hat{M}^{(j)} = \hat{1}$$
$$= \text{Tr}_E[\hat{U}(\hat{\rho} \otimes |0\rangle_{EE}\langle 0|)\hat{U}^\dagger]$$

$\hat{\rho}_{\text{out}} = \mathcal{C}(\hat{\rho})$ completely-positive trace-preserving map

CPTP map

Generalized measurement

This is exactly the most we can do !!

$$p_j = \mathrm{Tr}[\widehat{F}^{(j)}\widehat{\rho}] \ \text{ with } \sum_j \widehat{F}^{(j)} = \hat{1}$$

Examples

Orthogonal measurement on basis $\{|a_j\rangle\}$

$$\widehat{F}^{(j)} = |a_j\rangle\langle a_j|$$

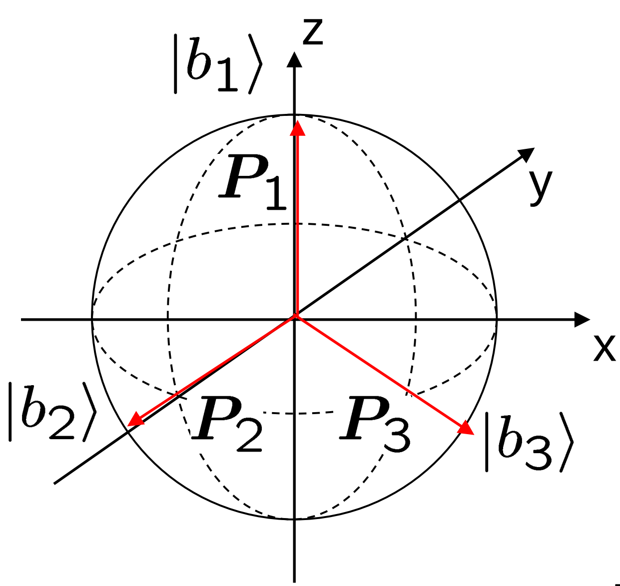
Trine measurement on a qubit

$$\widehat{F}^{(j)} = \tfrac{2}{3}|b_j\rangle\langle b_j|$$

$|b_j\rangle\langle b_j| = \frac{1}{2}\left(\hat{1} + \boldsymbol{P}_j \cdot \hat{\boldsymbol{\sigma}}\right)$

$\sum_j \boldsymbol{P}_j = 0$

$$\longrightarrow \sum_j \widehat{F}^{(j)} = \hat{1}$$



Universal NOT ? Spin reversal ?

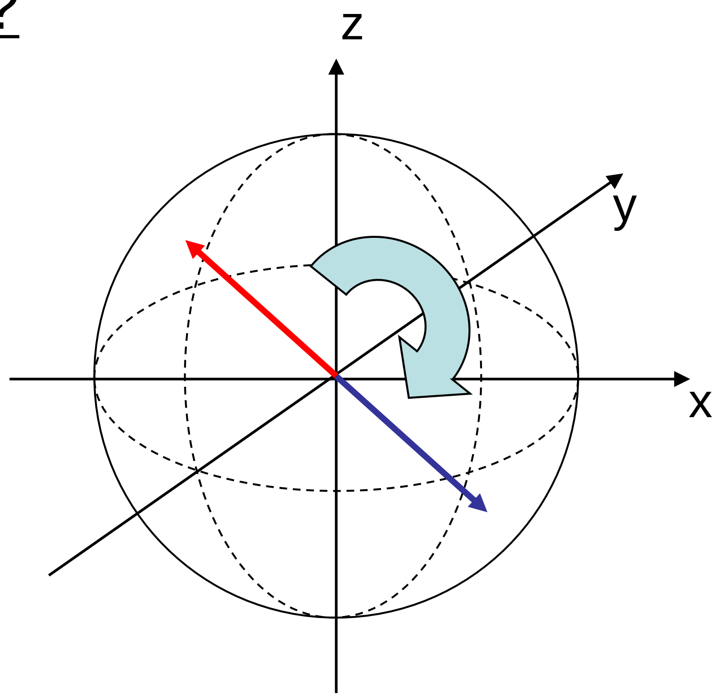
Bloch vector

$$\boldsymbol{P} \to -\boldsymbol{P}$$

linear map $\widehat{\rho} \to \mathcal{C}(\widehat{\rho})$

$$\mathcal{C}(\widehat{1}) = \widehat{1} \qquad \mathcal{C}(\widehat{\sigma}_x) = -\widehat{\sigma}_x$$

$$\mathcal{C}(\widehat{\sigma}_y) = -\widehat{\sigma}_y \qquad \mathcal{C}(\widehat{\sigma}_z) = -\widehat{\sigma}_z$$



$$\mathcal{C}(|0\rangle\langle 0|) = |1\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 1|) = |0\rangle\langle 0|$$

$$\mathcal{C}(|0\rangle\langle 1|) = -|0\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 0|) = -|1\rangle\langle 0|$$

$$\widehat{\sigma}_x = |1\rangle\langle 0| + |0\rangle\langle 1|$$

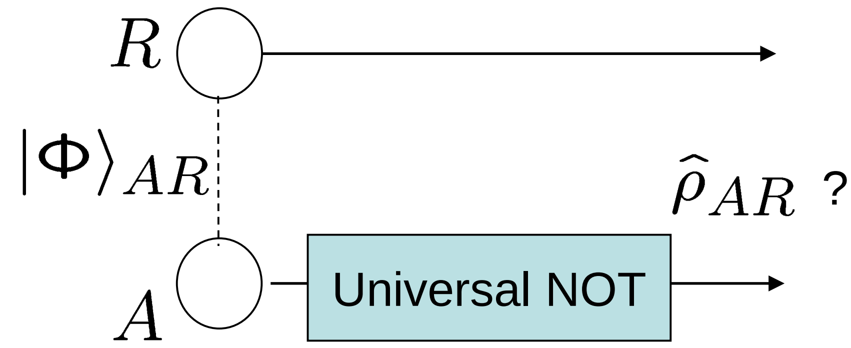
$$\widehat{\sigma}_y = i|1\rangle\langle 0| - i|0\rangle\langle 1|$$

$$\widehat{\sigma}_z = |0\rangle\langle 0| - |1\rangle\langle 1|$$

$$\widehat{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$$

This map is positive, but...

Universal NOT ? Spin reversal ?



$$\mathcal{C}(|0\rangle\langle 0|) = |1\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 1|) = |0\rangle\langle 0|$$

$$\mathcal{C}(|0\rangle\langle 1|) = -|0\rangle\langle 1|$$

$$\mathcal{C}(|1\rangle\langle 0|) = -|1\rangle\langle 0|$$

$$\begin{aligned} 2|\Phi\rangle\langle\Phi| &= (|00\rangle + |11\rangle)(\langle 00| + \langle 11|) \\ &= |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11| \end{aligned}$$

$$\begin{aligned} 2\hat{\rho}_{AR} \equiv 2(\mathcal{C}\otimes\mathcal{I})|\Phi\rangle\langle\Phi| &= \\ &= |10\rangle\langle 10| - |00\rangle\langle 11| - |11\rangle\langle 00| + |01\rangle\langle 01| \end{aligned}$$

$$2\hat{\rho}_{AR}(|00\rangle + |11\rangle) = -|11\rangle - |00\rangle = -(|00\rangle + |11\rangle)$$

$\hat{\rho}_{AR}$ has a negative eigenvalue! (The map is not completely positive.)

$\longrightarrow$ Universal NOT is impossible.

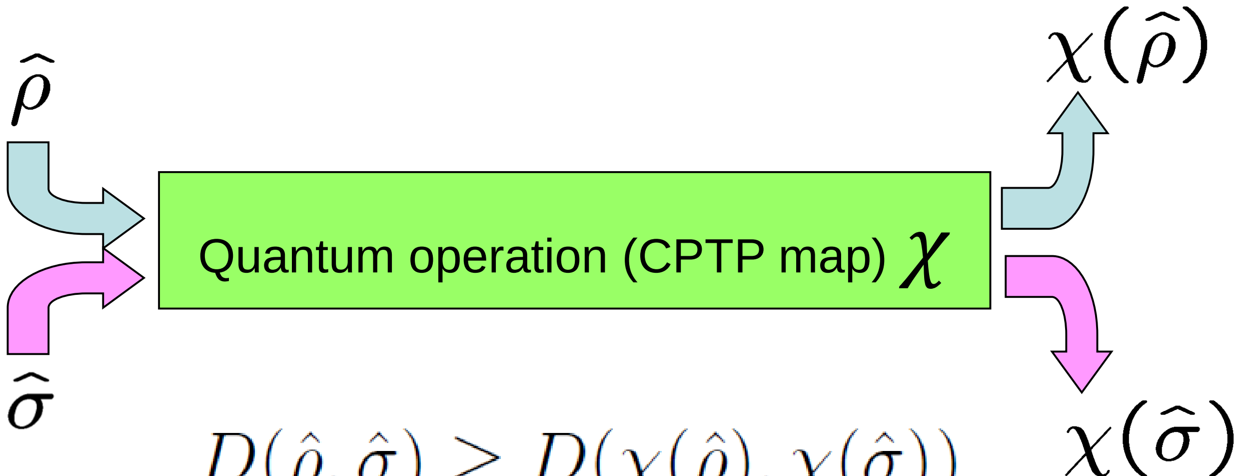
Distinguishability

Measure of distinguishability between two states $D(\hat{\rho}, \hat{\sigma})$

A quantity describing how we can distinguish between the two states in principle.

The distinguishability should never be improved by a quantum operation.

Monotonicity under quantum operations

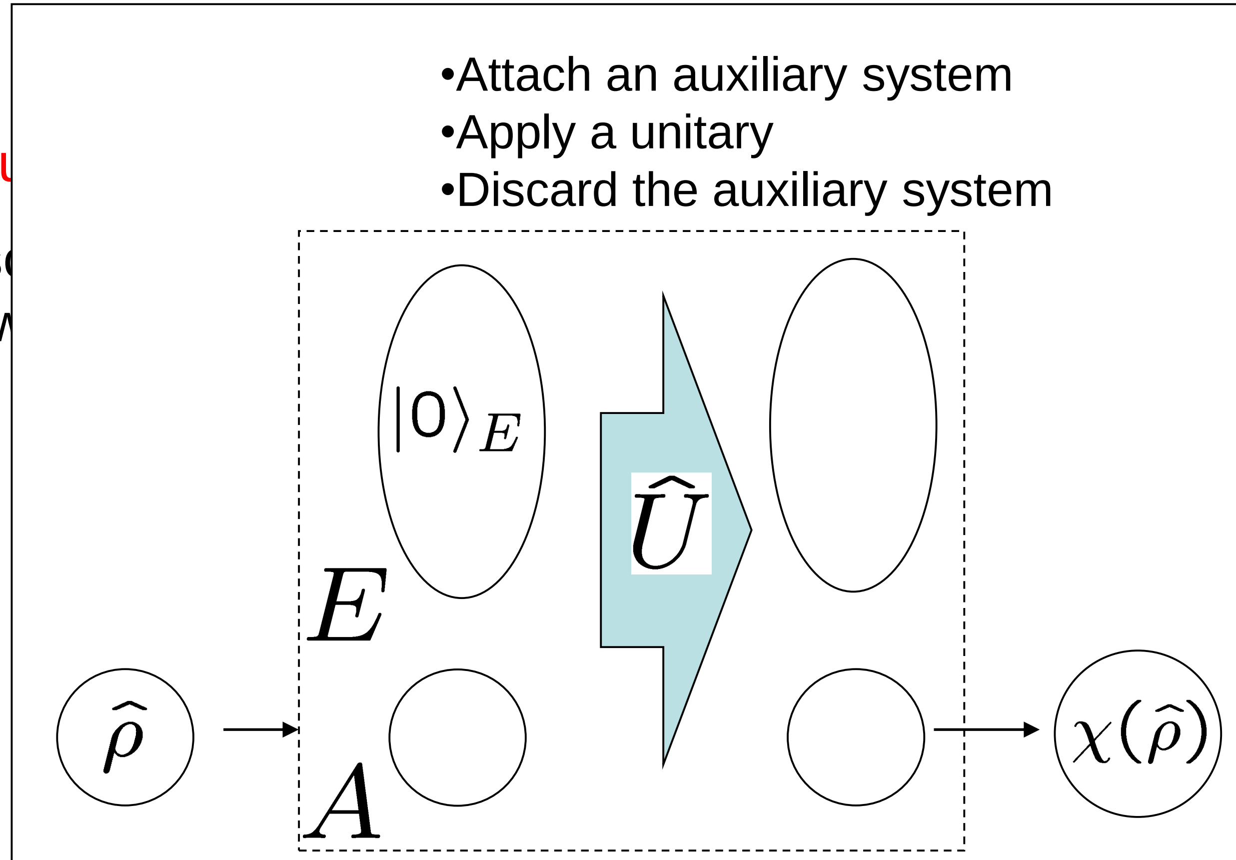


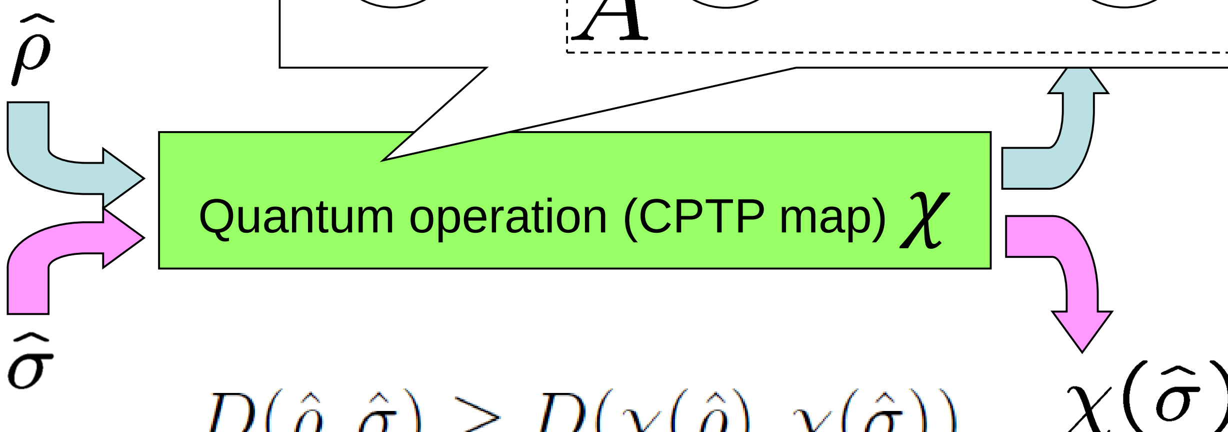
$$D(\hat{\rho}, \hat{\sigma}) \geq D(\chi(\hat{\rho}), \chi(\hat{\sigma}))$$

Distinguishability

Measure of distingu

A quantity desc
between the tw





$$D(\hat{\rho},\hat{\sigma}) \geq D(\chi(\hat{\rho}),\chi(\hat{\sigma}))$$

Fidelity

$F(\hat{\rho}, \hat{\sigma})$ Measure of indistinguishability between two states (closeness)

$F(\hat{\rho}, \hat{\sigma}) = 1$ The two states are identical

$F(\hat{\rho}, \hat{\sigma}) = 0$ The two states are perfectly distinguishable

$D(\hat{\rho}, \hat{\sigma}) = 1 - F(\hat{\rho}, \hat{\sigma})$ is a measure of distinguishability

$$D(\hat{\rho}, \hat{\sigma}) \geq D(\chi(\hat{\rho}), \chi(\hat{\sigma}))$$

$$F(\hat{\rho}, \hat{\sigma}) \leq F(\chi(\hat{\rho}), \chi(\hat{\sigma}))$$

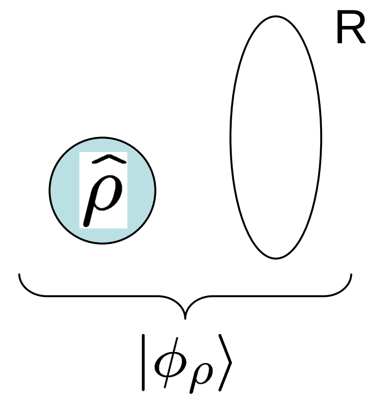
Fidelity

Definition

$$\boxed{F(\hat{\rho},\hat{\sigma}) \equiv \max |\langle \phi_\rho | \phi_\sigma \rangle|^2}$$

$$\mathrm{Tr}_R[|\phi_\rho\rangle\langle\phi_\rho|] = \hat{\rho}$$
$$\mathrm{Tr}_R[|\phi_\sigma\rangle\langle\phi_\sigma|] = \hat{\sigma}$$

(purifications)



Properties

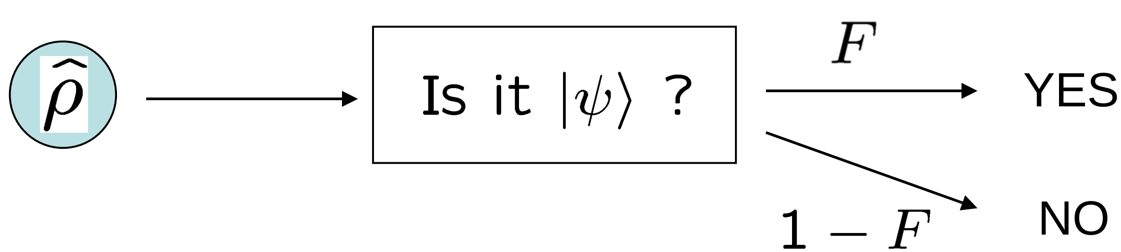
$$F(\hat{\rho},\hat{\sigma}) \equiv \max |\langle \phi_\rho | \phi_\sigma \rangle|^2 = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\hat{\sigma}}\hat{\rho}\sqrt{\hat{\sigma}}}\right)^2$$

$$F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|) = |\langle\phi|\psi\rangle|^2$$

Direct generalization of the magnitude of the inner product

$$F(\hat{\rho}, |\psi\rangle\langle\psi|) = \langle\psi|\hat{\rho}|\psi\rangle$$

Operational meaning of the fidelity



Fidelity

Definition

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$$\mathrm{Tr}_R[|\phi_\rho\rangle\langle\phi_\rho|] = \hat{\rho}$$
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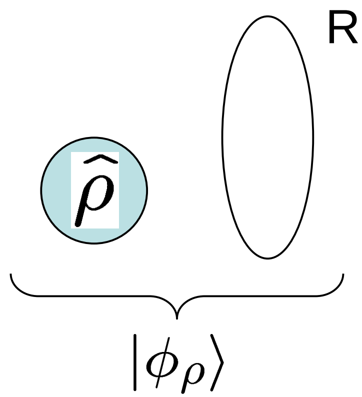
(purifications)



Properties

$$F(\hat{\rho},\hat{\sigma}) \equiv \max |\langle \phi_\rho | \phi_\sigma \rangle|^2 = \|\sqrt{\hat{\rho}}\sqrt{\hat{\sigma}}\|^2 = \left(\mathrm{Tr}\sqrt{\sqrt{\hat{\sigma}}\hat{\rho}\sqrt{\hat{\sigma}}}\right)^2$$

$$F(|\phi\rangle\langle\phi|, |\psi\rangle\langle\psi|) = |\langle\phi|\psi\rangle|^2$$

Direct generalization of the magnitude of the inner product

$$F(\hat{\rho}, |\psi\rangle\langle\psi|) = \langle\psi|\hat{\rho}|\psi\rangle$$

Operational meaning of the fidelity

(not applicable to general $F(\hat{\rho},\hat{\sigma})$)

$$F(\hat{\rho}_1 \otimes \hat{\rho}_2, \hat{\sigma}_1 \otimes \hat{\sigma}_2) = F(\hat{\rho}_1,\hat{\sigma}_1)F(\hat{\rho}_2,\hat{\sigma}_2)$$

Multiplicativity

Fidelity

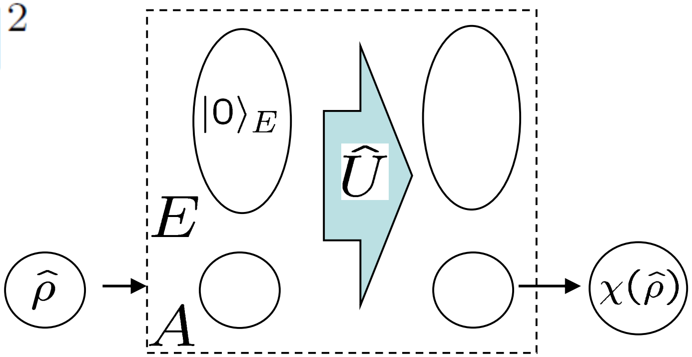
Proof of the monotonicity $F(\chi(\hat{\rho}),\chi(\hat{\sigma})) \geq F(\hat{\rho},\hat{\sigma})$

$$F(\hat{\rho},\hat{\sigma}) := \max|\langle\phi_\rho|\phi_\sigma\rangle|^2 \geq |\langle\phi_\rho|\phi_\sigma\rangle|^2$$

Attach an auxiliary system

Apply a unitary

Discard the auxiliary system



Fidelity

Proof of the monotonicity $F(\chi(\hat{\rho}),\chi(\hat{\sigma})) \geq F(\hat{\rho},\hat{\sigma})$

$$F(\hat{\rho},\hat{\sigma}) := \max |\langle\phi_\rho|\phi_\sigma\rangle|^2 \geq |\langle\phi_\rho|\phi_\sigma\rangle|^2$$

Attach an auxiliary system

Apply a unitary

Discard the auxiliary system

Consider purifications achieving $F(\hat{\rho},\hat{\sigma}) = |\langle\phi_\rho^*|\phi_\sigma^*\rangle|^2$

$$\hat{\rho} \otimes |0\rangle\langle 0| \quad \cdots\cdots \quad |\phi_\rho^*\rangle \otimes |0\rangle$$

$$\hat{\sigma} \otimes |0\rangle\langle 0| \quad \cdots\cdots \quad |\phi_\sigma^*\rangle \otimes |0\rangle$$

$$F(\hat{\rho} \otimes |0\rangle\langle 0|, \hat{\sigma} \otimes |0\rangle\langle 0|) \geq |\langle\phi_\rho^*|\phi_\sigma^*\rangle\langle 0|0\rangle|^2 = F(\hat{\rho},\hat{\sigma})$$

Fidelity

Proof of the monotonicity $F(\chi(\hat\rho),\chi(\hat\sigma)) \geq F(\hat\rho,\hat\sigma)$

$$F(\hat\rho,\hat\sigma) := \max |\langle\phi_\rho|\phi_\sigma\rangle|^2 \geq |\langle\phi_\rho|\phi_\sigma\rangle|^2$$

Attach an auxiliary system

Apply a unitary

Discard the auxiliary system

Consider purifications achieving $F(\hat\rho,\hat\sigma) = |\langle\phi_\rho^*|\phi_\sigma^*\rangle|^2$

$$\hat U\hat\rho\hat U^\dagger \quad \cdots\cdots \quad (\hat U\otimes\hat 1)|\phi_\rho^*\rangle$$

$$\hat U\hat\sigma\hat U^\dagger \quad \cdots\cdots \quad (\hat U\otimes\hat 1)|\phi_\sigma^*\rangle$$

$$F(\hat U\hat\rho\hat U^\dagger,\hat U\hat\sigma\hat U^\dagger) \geq |\langle\phi_\rho^*|(\hat U\otimes\hat 1)^\dagger(\hat U\otimes\hat 1)|\phi_\sigma^*\rangle|^2 = F(\hat\rho,\hat\sigma)$$

Fidelity

Proof of the monotonicity $F(\chi(\hat{\rho}), \chi(\hat{\sigma})) \geq F(\hat{\rho}, \hat{\sigma})$

$$F(\hat{\rho}, \hat{\sigma}) := \max |\langle \phi_\rho | \phi_\sigma \rangle|^2 \geq |\langle \phi_\rho | \phi_\sigma \rangle|^2$$

Attach an auxiliary system

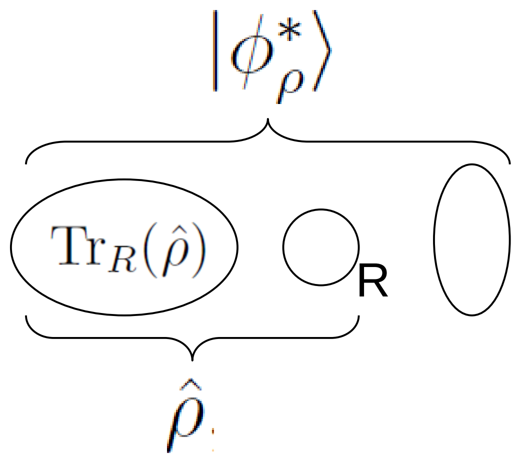
Apply a unitary

Discard the auxiliary system

Consider purifications achieving $F(\hat{\rho}, \hat{\sigma}) = |\langle \phi^*_\rho | \phi^*_\sigma \rangle|^2$

$$\mathrm{Tr}_R(\hat{\rho}) \cdots\cdots |\phi^*_\rho\rangle$$

$$\mathrm{Tr}_R(\hat{\sigma}) \cdots\cdots |\phi^*_\sigma\rangle$$



$$F(\mathrm{Tr}_R(\hat{\rho}), \mathrm{Tr}_R(\hat{\sigma})) \geq |\langle \phi^*_\rho | \phi^*_\sigma \rangle|^2 = F(\hat{\rho}, \hat{\sigma})$$

No-cloning theorem

$$F(\hat{\rho}_1 \otimes \hat{\rho}_2, \hat{\sigma}_1 \otimes \hat{\sigma}_2) = F(\hat{\rho}_1, \hat{\sigma}_1) F(\hat{\rho}_2, \hat{\sigma}_2)$$ Multiplicativity

$$F(\hat{\rho}, \hat{\sigma}) \le F(\chi(\hat{\rho}), \chi(\hat{\sigma}))$$ Monotonicity

Is it possible to realize $\begin{aligned} \chi(\hat{\rho}) &= \hat{\rho} \otimes \hat{\rho} \\ \chi(\hat{\sigma}) &= \hat{\sigma} \otimes \hat{\sigma} \end{aligned}$?

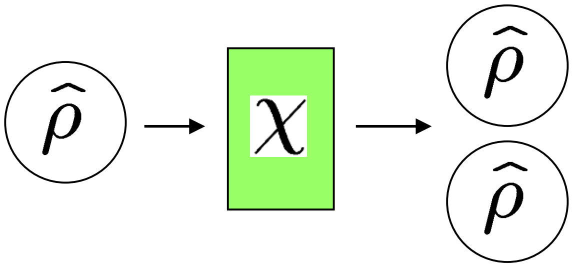
$$F(\hat{\rho}, \hat{\sigma}) \le F(\chi(\hat{\rho}), \chi(\hat{\sigma})) = F(\hat{\rho} \otimes \hat{\rho}, \hat{\sigma} \otimes \hat{\sigma}) = F(\hat{\rho}, \hat{\sigma})^2$$

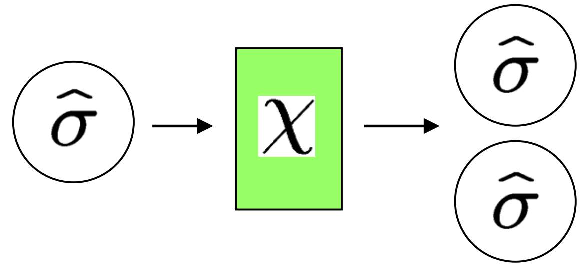
Possible only when $F(\hat{\rho}, \hat{\sigma}) = 0$ or 1

It is impossible to create independent copies of two inputs that are neither distinguishable nor identical.

No-cloning theorem for classical case?

It is impossible to create independent copies of two inputs that are neither distinguishable nor identical.





If we allow mixed states, partial distinguishability is not rare even in classical states.

$$\hat{\rho} = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \qquad \hat{\sigma} = \frac{1}{3}|0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1|$$

It is possible to create correlated copies. (Broadcasting)

$$\chi(\hat{\rho}) = \frac{2}{3}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

$$\chi(\hat{\sigma}) = \frac{1}{3}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{2}{3}|1\rangle\langle 1| \otimes |1\rangle\langle 1|$$

The marginal states are the same as the input.

No-cloning theorem for pure states

It is impossible to create independent copies of two inputs that are neither distinguishable nor identical.

If the marginal state is pure, the subsystem has no correlation to other systems.

It is impossible to create copies of two nonorthogonal and nonidentical pure states.

Of course, it is impossible to create copies of unknown pure states.

What is peculiar about quantum mechanics?

Partially distinguishable $\longrightarrow$ No independent copies

Pure $\longrightarrow$ No correlation

These implications are not unique to quantum mechanics.

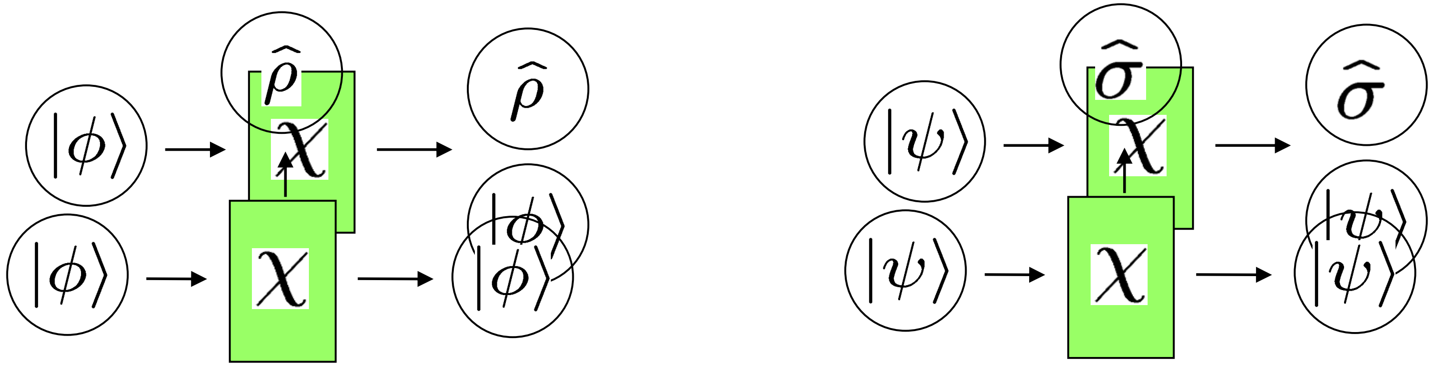
In quantum mechanics, there are cases where states are partially distinguishable and pure.

Partially distinguishable
and
Pure

No copies

Information-disturbance tradeoff

Suppose that $|\langle\phi|\psi\rangle| > 0$



$$|\langle\phi|\psi\rangle|^2 \le F(|\phi\rangle\langle\phi| \otimes \hat{\rho}, |\psi\rangle\langle\psi| \otimes \hat{\sigma})$$

$$= |\langle\phi|\psi\rangle|^2 F(\hat{\rho}, \hat{\sigma})$$

$$\longrightarrow F(\hat{\rho}, \hat{\sigma}) = 1 \qquad \hat{\rho} = \hat{\sigma}$$

If a process causes absolutely no disturbance on two nonorthogonal states, it leaves no trace about which of the states has been fed to the input.

Basic principle for a quantum cryptography scheme, called B92 protocol.

4. Communication resources

Classical channel

Quantum channel

Entanglement

- How does the state evolve under LOCC?
- Properties of maximally entangled states
- Bell basis

Quantum dense coding

Quantum teleportation

- Entanglement swapping

Resource conversion protocols and bounds

Classical channel

0
1 $\dashrightarrow$ 0
1

Ideal classical channel: faithful transfer of any signal chosen from d symbols

Parallel use of channels

0 1	$\dashrightarrow$	0 1			
				0a 0b 0c 1a 1b 1c	$\dashrightarrow$	0a 0b 0c 1a 1b 1c
a b c	$\dashrightarrow$	a b c			

d-symbol ideal classical channel
d'-symbol ideal classical channel $\Big\}$ (dd')-symbol ideal classical channel

Measure of usefulness

d-symbol ideal classical channel $\longrightarrow$ (log d) bits

Additive for ideal channels

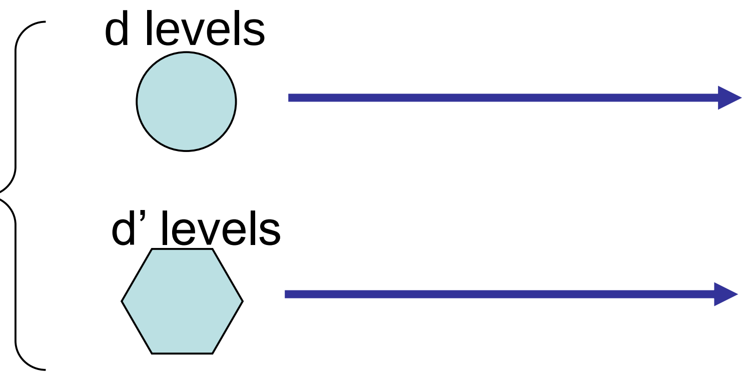
Quantum channel

$\alpha|0\rangle + \beta|1\rangle$ $\qquad\qquad\qquad\qquad$ $\alpha|0\rangle + \beta|1\rangle$

Ideal quantum channel: faithful transfer of any state of an d-level system
(Hilbert space of dimension d)

Parallel use of channels

(dd')-level system



Measure of usefulness

d-level ideal quantum channel $\longrightarrow$ (log d) qubits

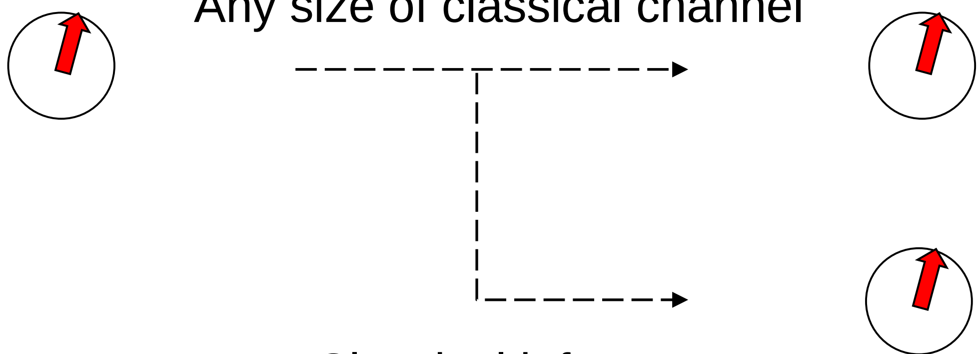
Additive for ideal channels

Can classical channels substitute a quantum channel?

NO (with no other resources)

Suppose that it was possible …

Any size of classical channel



Classical info
can be copied

The same procedure should
result in the same state.

This amounts to the cloning of unknown quantum states,
which is forbidden.

Can a quantum channel substitute a classical channel?

Of course yes.

But not so bizarre (with no other resources).

(without use of other communication resources)

n-qubit ideal quantum channel can only substitute a n-bit classical channel.

(Holevo bound)

Suppose that transfer of an d-level system can convey any signal from s symbols faithfully.

Measurement

$$j = 1, 2, \ldots, s \qquad \hat{\rho}_j \longrightarrow \hat{\rho}_j \longrightarrow j'$$

$$\dim \mathcal{H} = d$$

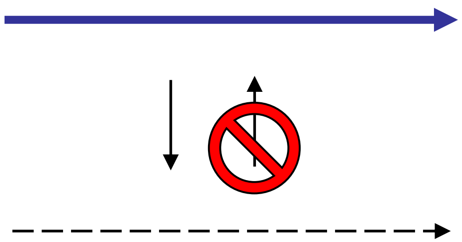
Always $j' = j$

Recall that any measurement must be described by a POVM. $\sum_{j'} \hat{F}_{j'} = \hat{1}$

$$\mathrm{Tr}(\hat{F}_j \hat{\rho}_j) = 1$$

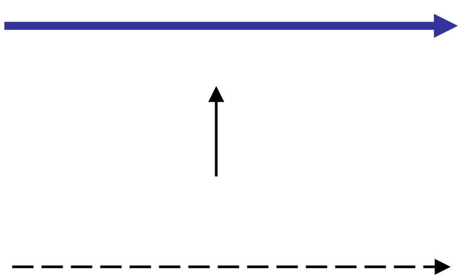
$$s = \sum_j \mathrm{Tr}(\hat{F}_j \hat{\rho}_j) \le \sum_j \mathrm{Tr}(\hat{F}_j \hat{1}) = \sum_j \mathrm{Tr}(\hat{F}_j) \le \sum_{j'} \mathrm{Tr}(\hat{F}_{j'}) = \mathrm{Tr}(\hat{1}) = d$$

Difference between quantum and classical channels



We have seen that a quantum channel is more powerful than a classical channel.

Can we pin down what is missing in a classical channel?



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Other resource

Entanglement

I've already bought a classical channel, but now I want to use a quantum channel. Do I have to buy the quantum channel?

Oh, you can buy this optional package for a cheaper price, and upgrade the classical channel to a quantum channel!

Maximally entangled states

$$\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$$

Orthonormal bases $\{|k\rangle_A\}_{k=1,2,\ldots,d}$ $\{|k\rangle_B\}_{k=1,2,\ldots,d}$

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Maximally entangled state

Marginal states

$$\hat{\rho}_A = d^{-1}\hat{1}_A \qquad \hat{\rho}_B = d^{-1}\hat{1}_B$$

Maximally mixed state Maximally mixed state

Maximally entangled states (MES)

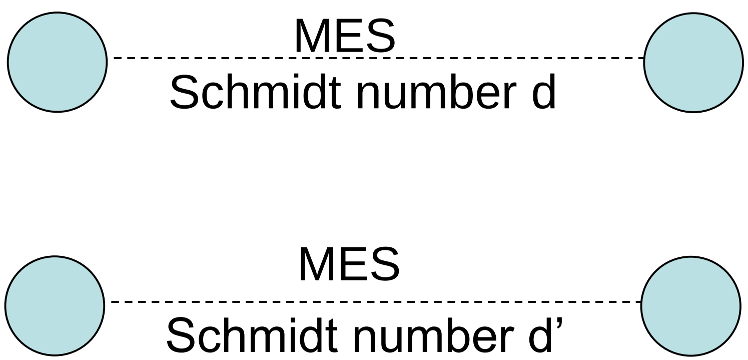
"ideal" entangled states

$$\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Rank of $\hat{\rho}_A$ = Rank of $\hat{\rho}_B$ = d

(Schmidt number)

Putting two MESs together



MES with

Schmidt number dd'

$$\left(\sum_{j=1}^{d} \frac{1}{\sqrt{d}} |j\rangle_A \otimes |j\rangle_B\right) \otimes \left(\sum_{k=1}^{d'} \frac{1}{\sqrt{d'}} |k\rangle_{A'} \otimes |k\rangle_{B'}\right) = \sum_{j,k} \frac{1}{\sqrt{dd'}} |jk\rangle_{AA'} \otimes |jk\rangle_{BB'}$$

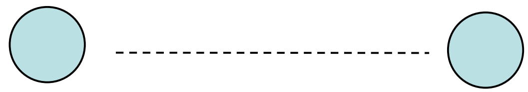
Measure of entanglement

MES with Schmidt number d $\longrightarrow$ (log d) ebits

Additive for MESs

Maximally entangled states (MES)

Schmidt number d=2



$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

= 1 ebit

$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B)$$

$$\frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$$

$$\frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B)$$

Can we increase entanglement by classical communication?

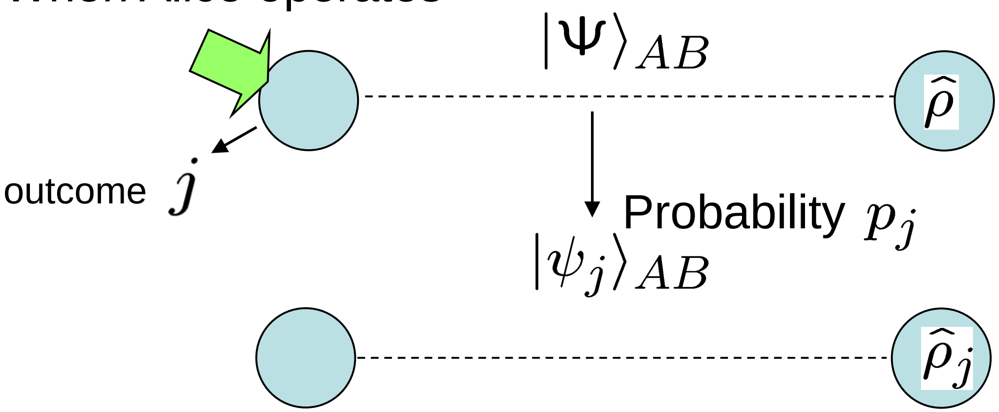
Unlimited use of classical communication:

LOCC: Local operations and classical communication

Any LOCC procedure can be made a sequential one:

- Alice applies local operations
- Alice communicates to Bob
- Bob applies local operations
- Bob communicates to Alice
- Alice …..

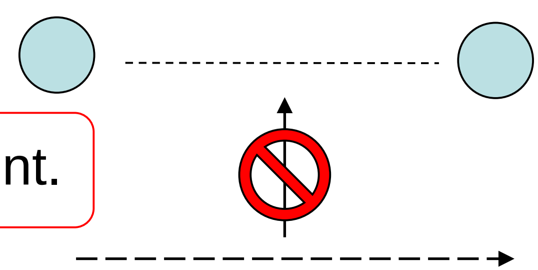
When Alice operates



$$\boxed{\sum_j p_j \hat{\rho}_j = \hat{\rho}}$$

$$\mathrm{Rank}\ \hat{\rho} \geq \mathrm{Rank}\ \hat{\rho}_j$$

Schmidt number never increases under LOCC (even probabilistically)

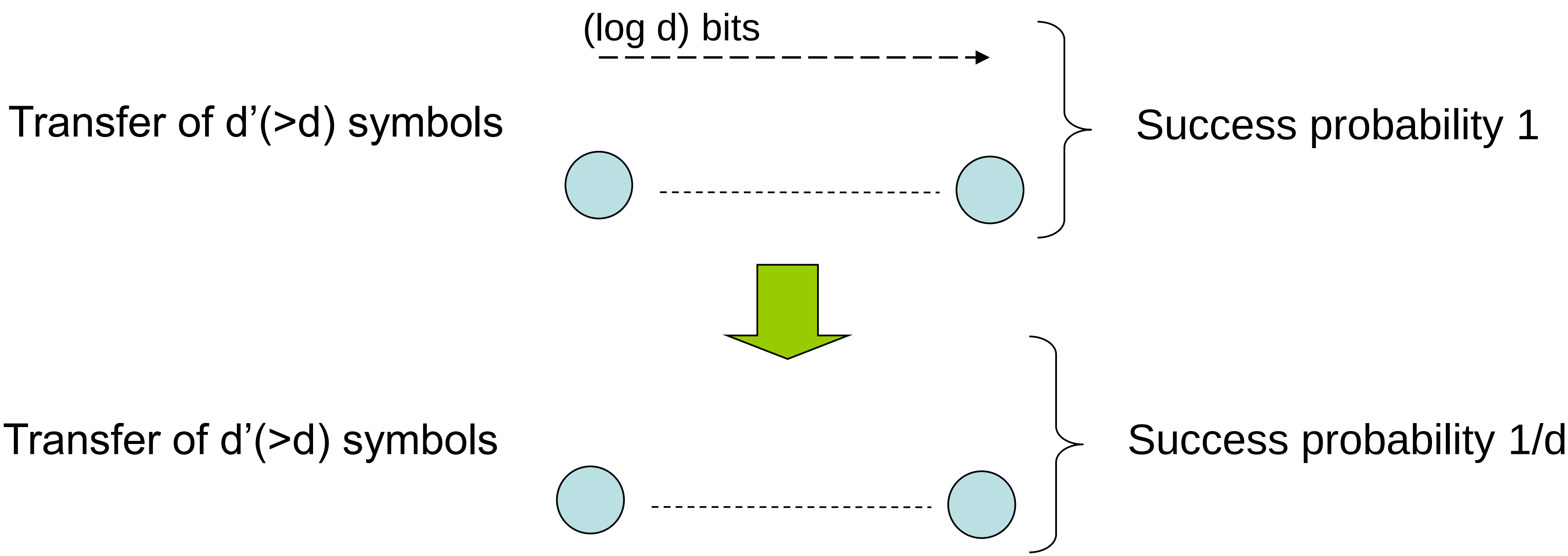


Classical channels cannot increase entanglement.

Can entanglement increase classical communication?

d-symbol ideal classical channel

The outcome can be correctly predicted with probability at least 1/d.



Can entanglement increase classical communication?

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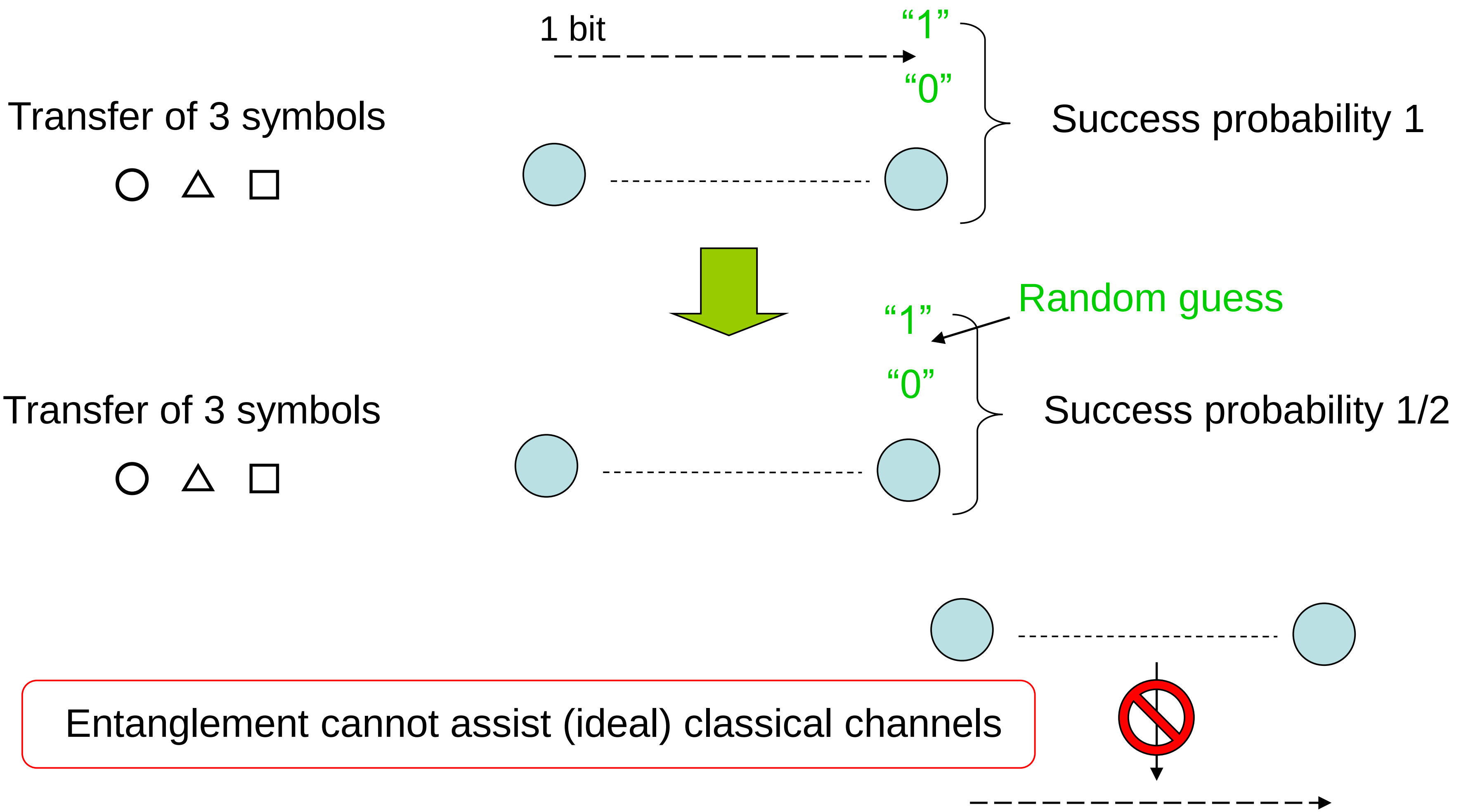
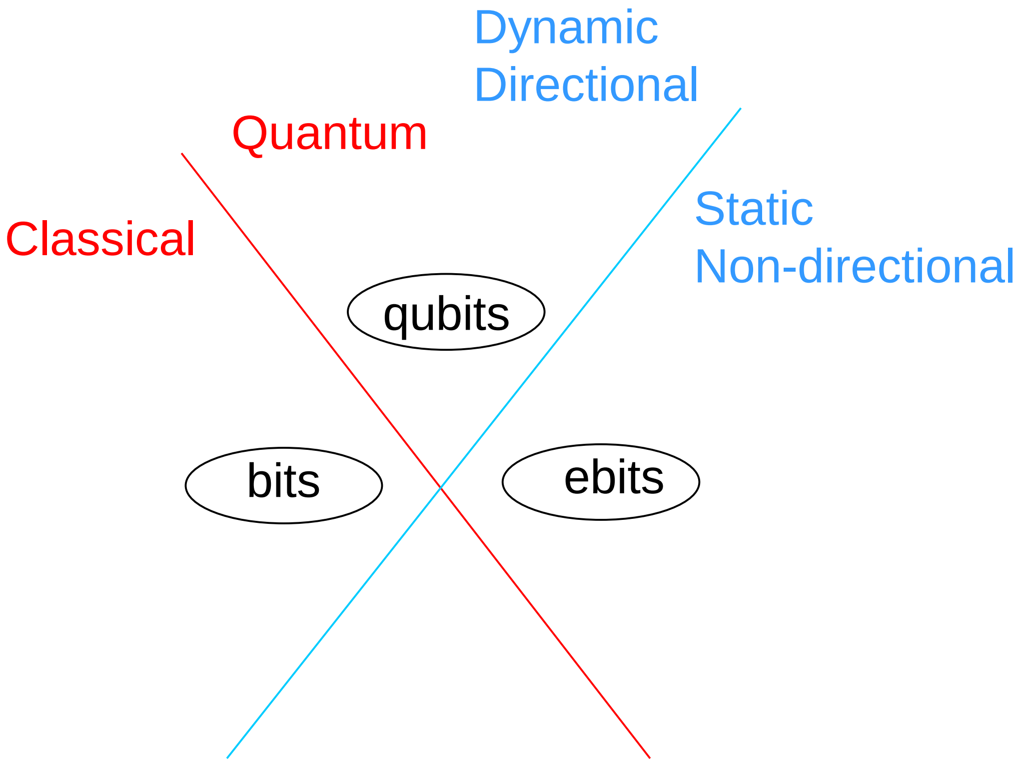


Entanglement cannot assist (ideal) classical channels

Communication resources



Restrictions

bits alone → no ebits

ebits alone → no bits

1 qubit alone → no more than 1 bit

Resource conversion protocols

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

Conversion to bits

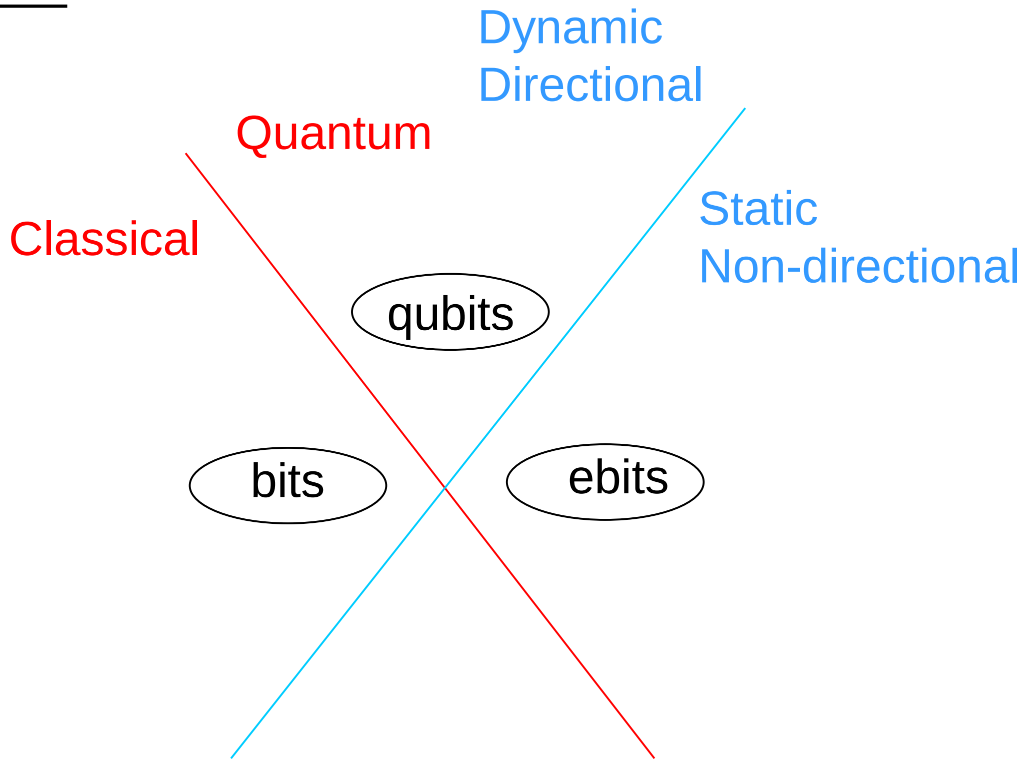
Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit



Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit

Properties of maximally entangled states

$$|\Phi\rangle_{AB} = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B$$

Pair of local states (relative states)

$$\boxed{\frac{1}{\sqrt{d}} |\phi\rangle_A = {}_B\langle\phi^*||\Phi\rangle_{AB}}$$

$$|\phi\rangle_A = \sum_k \alpha_k |k\rangle_A$$

A B

$$|\phi^*\rangle_B = \sum_k \overline{\alpha_k} |k\rangle_B$$

measurement

$p = 1/d$

Locally maximally mixed

$$\boxed{\hat{\rho}_A = \mathrm{Tr}_B |\Phi\rangle\langle\Phi| = \frac{1}{d}\hat{1}_A}$$

Convertibility via local unitary

$$\boxed{|\Phi'\rangle_{AB} = (\hat{1}_A \otimes \hat{U}_B)|\Phi\rangle_{AB}}$$

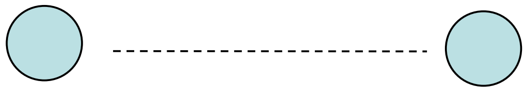
Orthonormal basis (Bell basis)

$$\boxed{\langle\Phi_j|\Phi_k\rangle = \delta_{jk} \; (j, k = 1, \ldots d^2)}$$

There exists an orthonormal basis composed of MESs.

Bell basis for a pair of qubits

$(d = 2)$



$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$$

$$\frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B)$$

$$\frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B + |0\rangle_A|1\rangle_B)$$

$$\frac{1}{\sqrt{2}}(|1\rangle_A|0\rangle_B - |0\rangle_A|1\rangle_B)$$

Bell basis

$\beta \equiv \exp[2\pi i/d] \quad (\beta^d = \beta^0 = 1, \beta^{-1} = \overline{\beta})$

Basis $\{|0\rangle, |1\rangle, \ldots, |d-1\rangle\} \quad (|d\rangle = |0\rangle)$

$$\hat{X} \equiv \sum_{j=0}^{d-1} |j+1\rangle\langle j| \qquad \hat{Z} \equiv \sum_{j=0}^{d-1} \beta^j |j\rangle\langle j| \qquad \text{(Unitary)}$$

$$\hat{X}^T = \hat{X}^{-1} \qquad \hat{Z}^T = \hat{Z}$$

$\hat{Z}^d = \hat{X}^d = \hat{1}$ Eigenvalues: $1, \beta, \beta^2, \ldots, \beta^{d-1}$

$$\hat{Z}\hat{X} = \beta \hat{X}\hat{Z} \qquad \hat{Z}^m \hat{X}^l = \beta^{lm} \hat{X}^l \hat{Z}^m$$

$$|\Phi_{0,0}\rangle \equiv \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_A \otimes |k\rangle_B \qquad \begin{aligned} (\hat{X}_A \otimes \hat{X}_B)|\Phi_{0,0}\rangle &= |\Phi_{0,0}\rangle \\ (\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{0,0}\rangle &= |\Phi_{0,0}\rangle \end{aligned}$$

Bell basis: $\{|\Phi_{l,m}\rangle\}$ $(l = 0, 1, \ldots, d-1;\ m = 0, 1, \ldots, d-1)$

$$|\Phi_{l,m}\rangle \equiv (\hat{X}_A^l \otimes \hat{Z}_B^m)|\Phi_{0,0}\rangle$$

$$\left.\begin{aligned} (\hat{X}_A \otimes \hat{X}_B)|\Phi_{l,m}\rangle &= \beta^{-m}|\Phi_{l,m}\rangle \\ (\hat{Z}_A \otimes \hat{Z}_B^{-1})|\Phi_{l,m}\rangle &= \beta^{l}|\Phi_{l,m}\rangle \end{aligned}\right\} \longrightarrow \text{All states are orthogonal.}$$

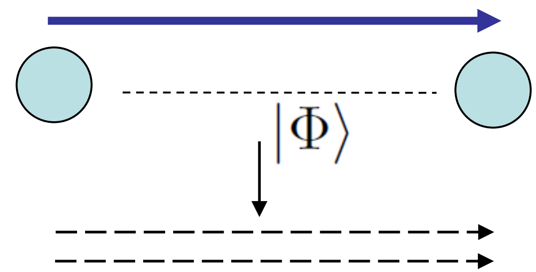
Quantum dense coding

$$|\Phi\rangle = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle \otimes |k\rangle$$

1 qubit + 1 ebit $\longrightarrow$ 2 bits

n qubits + n ebits $\longrightarrow$ 2n bits

(Dimension d) + (Schmidt number d)
$\rightarrow$ (d^2 symbols)

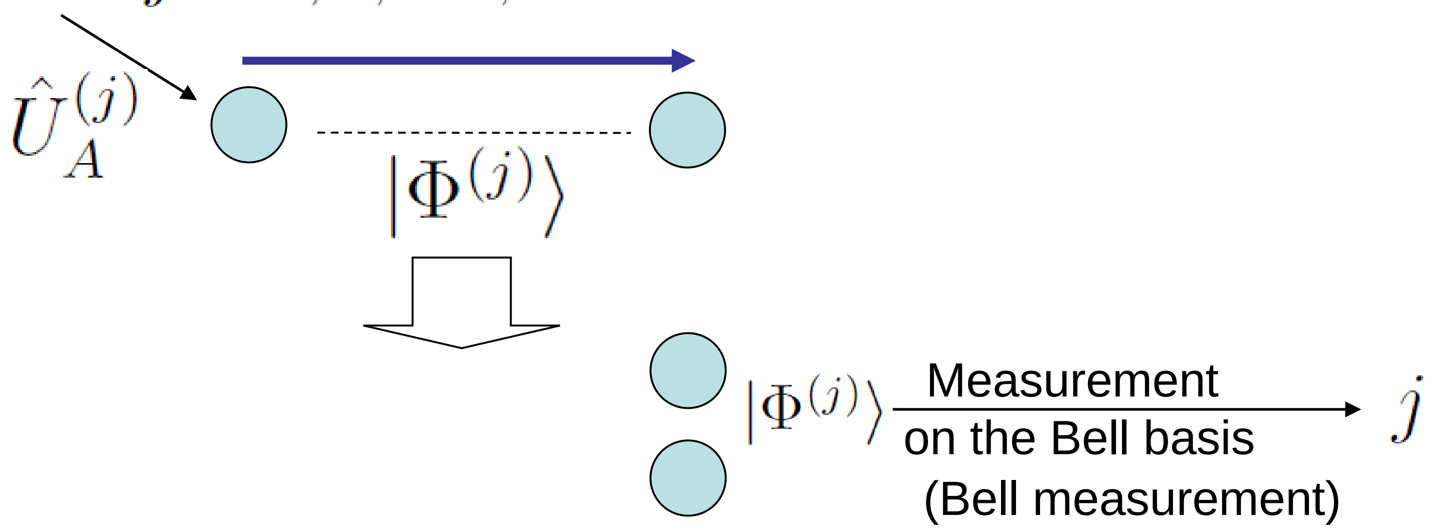


MES

Orthonormal basis (Bell basis) $\left\{ |\Phi^{(j)}\rangle \right\}_{j=1,2,\dots,d^2}$

Convertibility via local unitary $|\Phi^{(j)}\rangle = (\hat{U}_A^{(j)} \otimes \hat{1}_B)|\Phi\rangle$

d^2 symbols $j = 1, 2, \dots, d^2$



Resource conversion protocols

Conversion to ebits

Entanglement sharing

1 qubit $\longrightarrow$ 1 ebit

Conversion to bits

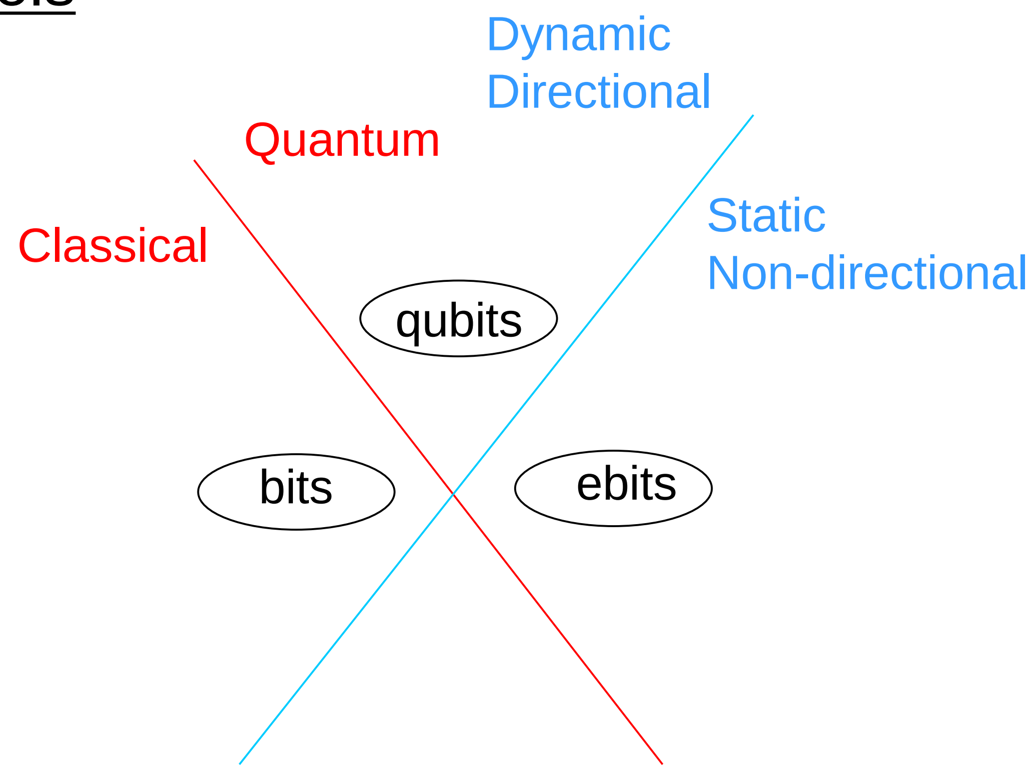
Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit $\longrightarrow$ 1 qubit



Restrictions

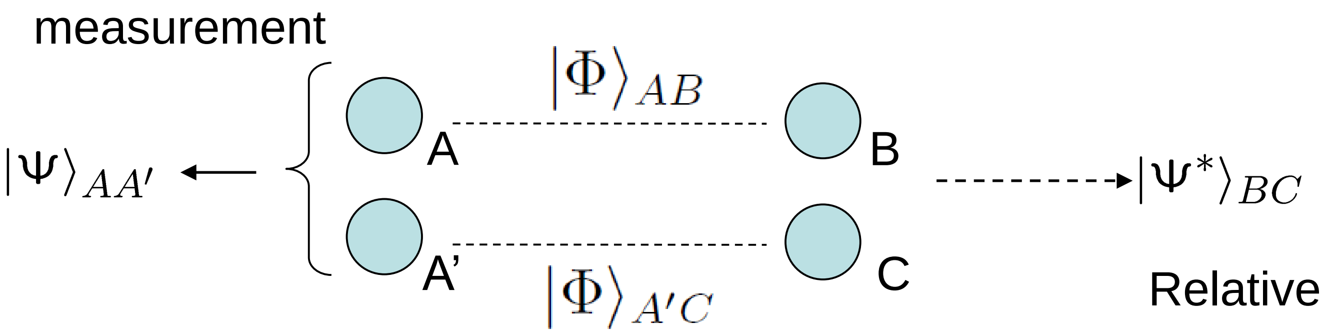
bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

1 qubit alone $\longrightarrow$ no more than 1 bit

Creating entanglement by nonlocal measurement

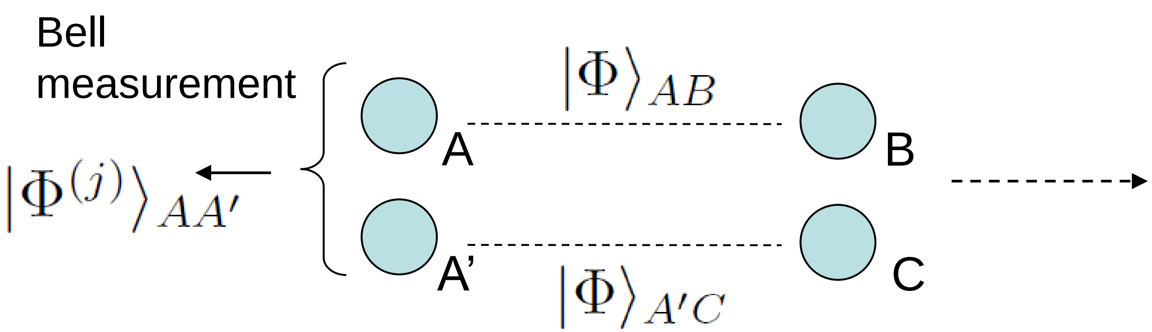
$$|\Phi\rangle = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle \otimes |k\rangle$$



Relative state of $|\Psi\rangle_{AA'}$

Same entanglement

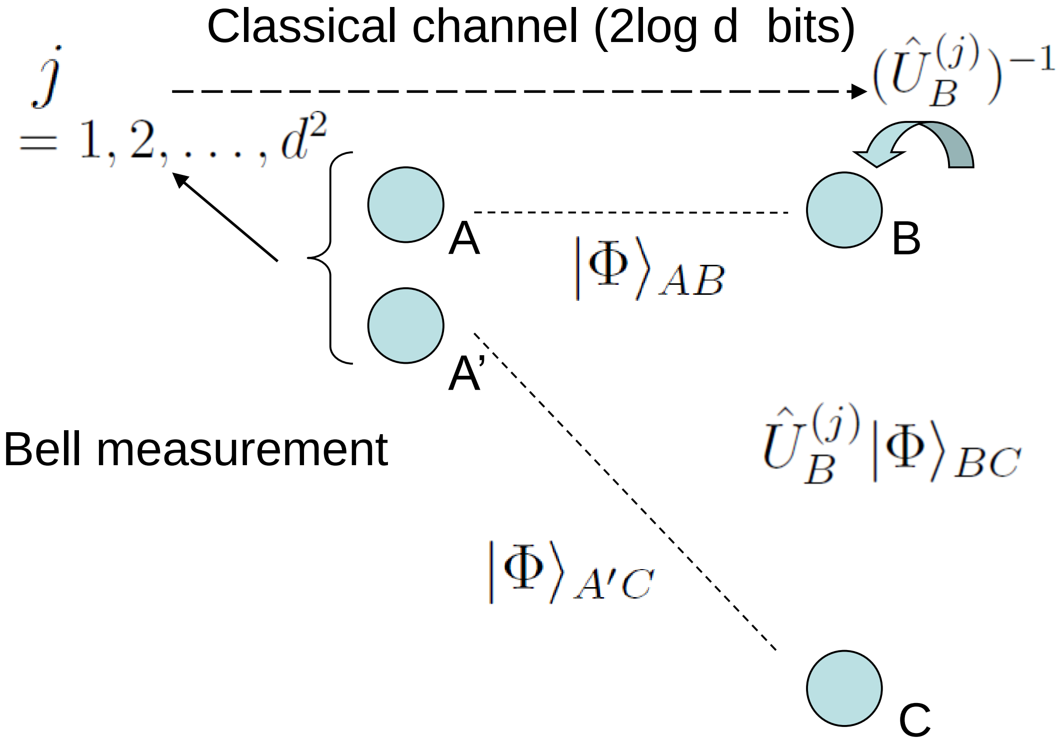
$$\left(\sum_{j=1}^{d} \frac{1}{\sqrt{d}} |j\rangle_A \otimes |j\rangle_B\right) \otimes \left(\sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle_{A'} \otimes |k\rangle_{B'}\right) = \sum_{j,k} \frac{1}{\sqrt{d^2}} |jk\rangle_{AA'} \otimes |jk\rangle_{BB'}$$

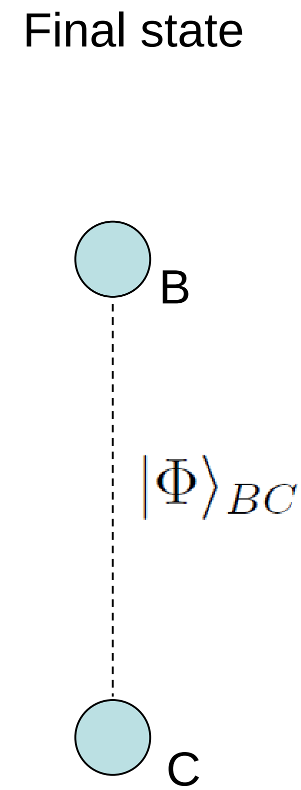


$$|\Phi^{(j)*}\rangle_{BC} = \hat{U}_B^{(j)} |\Phi\rangle_{BC}$$

Entanglement swapping

$$|\Phi\rangle = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle \otimes |k\rangle$$





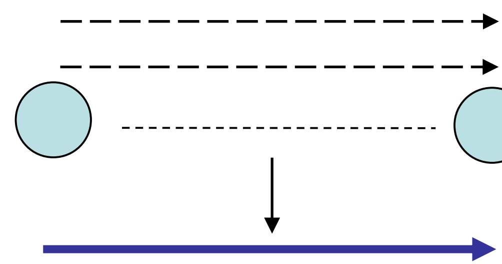
Quantum teleportation

1 ebit + 2 bit ⟶ 1 qubit

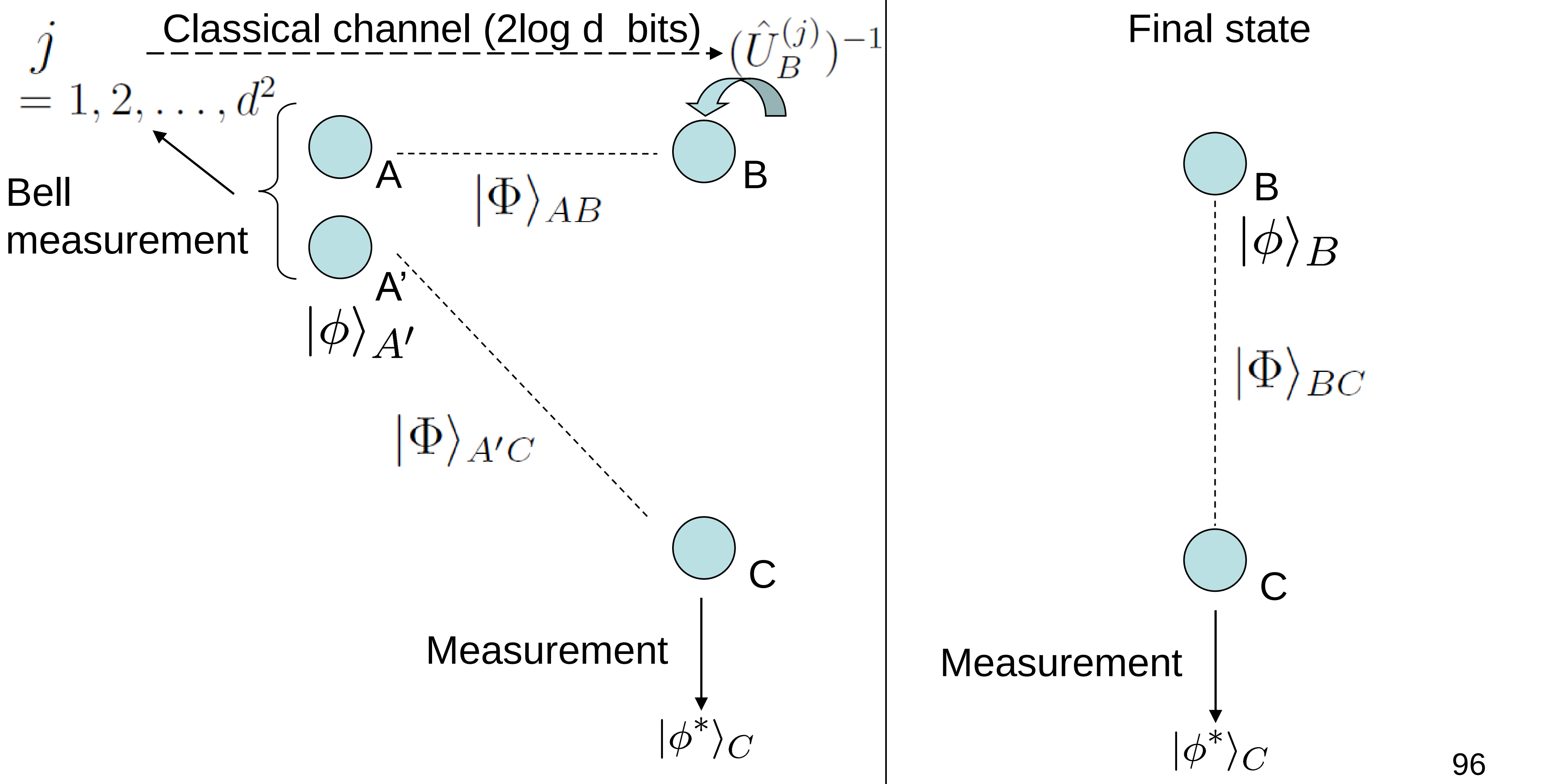
n ebits + 2n bits ⟶ n qubits

(d^2 symbols) + (Schmidt number d)

$\rightarrow$ (Dimension d)

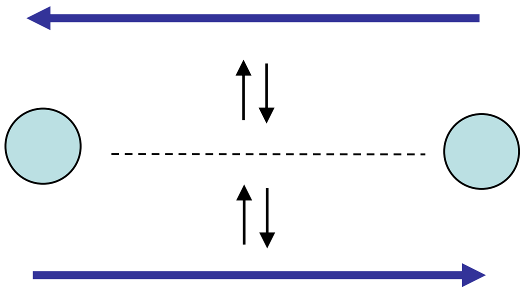


$$|\Phi\rangle = \sum_{k=1}^{d} \frac{1}{\sqrt{d}} |k\rangle \otimes |k\rangle$$



Quantum teleportation

If the cost of classical communication is neglected …



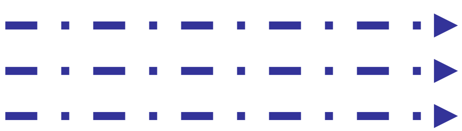
One can reserve the quantum channel by storing a quantum state.

One can use a quantum channel in the opposite direction.

A convenient way of quantum error correction (failure $\longrightarrow$ retry).

Noisy quantum channel

$|\phi\rangle$



$\hat{\rho} \to |\phi\rangle$

Recovering

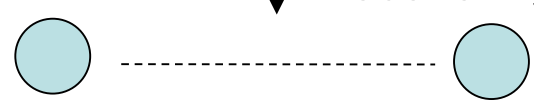
Failure $\longrightarrow$ no recovery.

Noisy quantum channel

$\downarrow$

Noisy entanglement

$\downarrow$ Recovering



Resource conversion protocols

Conversion to ebits

Entanglement sharing

1 qubit ⟶ 1 ebit

Conversion to bits

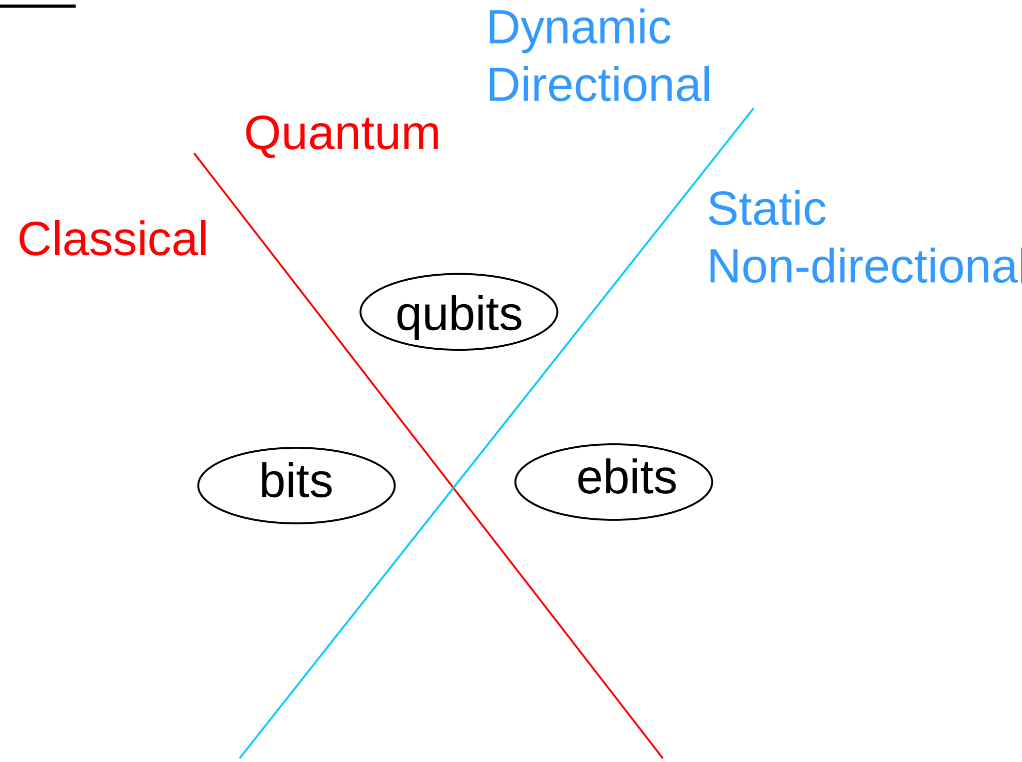
Quantum dense coding

1 qubit + 1 ebit ⟶ 2 bits

Conversion to qubits

Quantum teleportation

2 bits + 1 ebit ⟶ 1 qubit



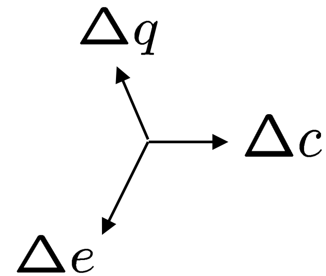
Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit

Resource conversion protocols and bounds



We can do the following…

Conversion to ebits

Entanglement sharing

1 qubit $\longrightarrow$ 1 ebit

$$(\Delta q, \Delta e, \Delta c) = (-1, 1, 0)$$

Conversion to bits

Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits

$$(\Delta q, \Delta e, \Delta c) = (-1, -1, 2)$$

Conversion to qubits

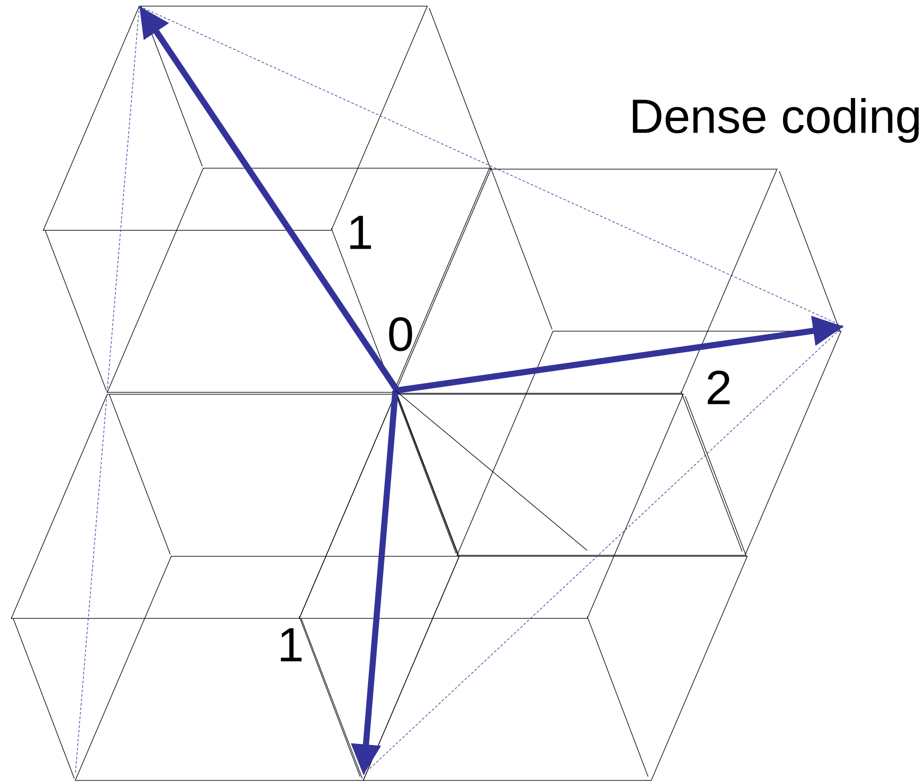
Quantum teleportation

2 bits + 1 ebit $\longrightarrow$ 1 qubit

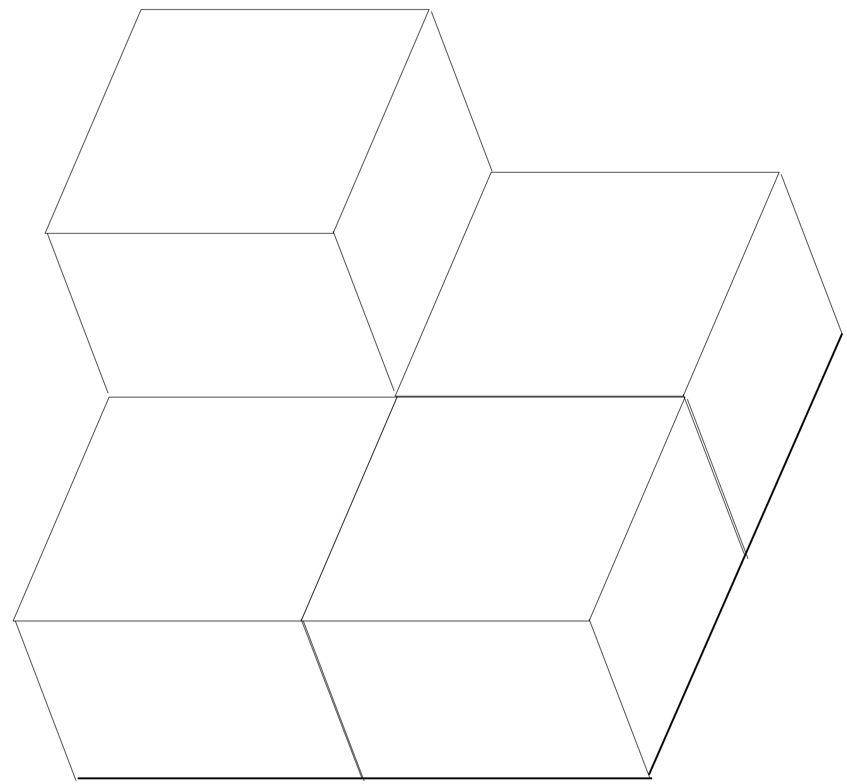
$$(\Delta q, \Delta e, \Delta c) = (1, -1, -2)$$

Teleportation

Dense coding



Entanglement sharing



Resource conversion protocols and bounds

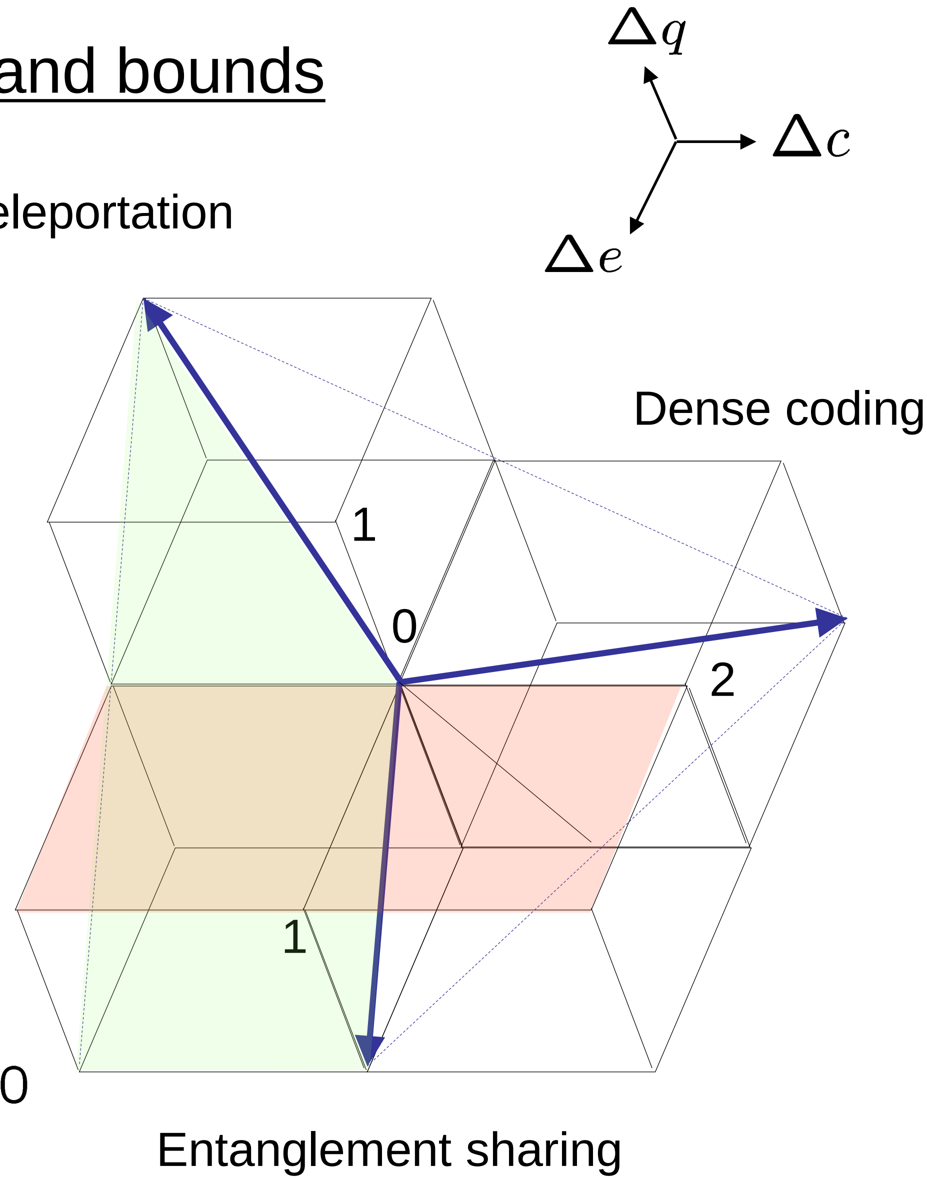
We can do the following…

Restrictions

bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

1 qubit alone $\longrightarrow$ no more than 1 bit



$$\Delta e + \Delta q \leq 0$$

Resource conversion protocols and bounds

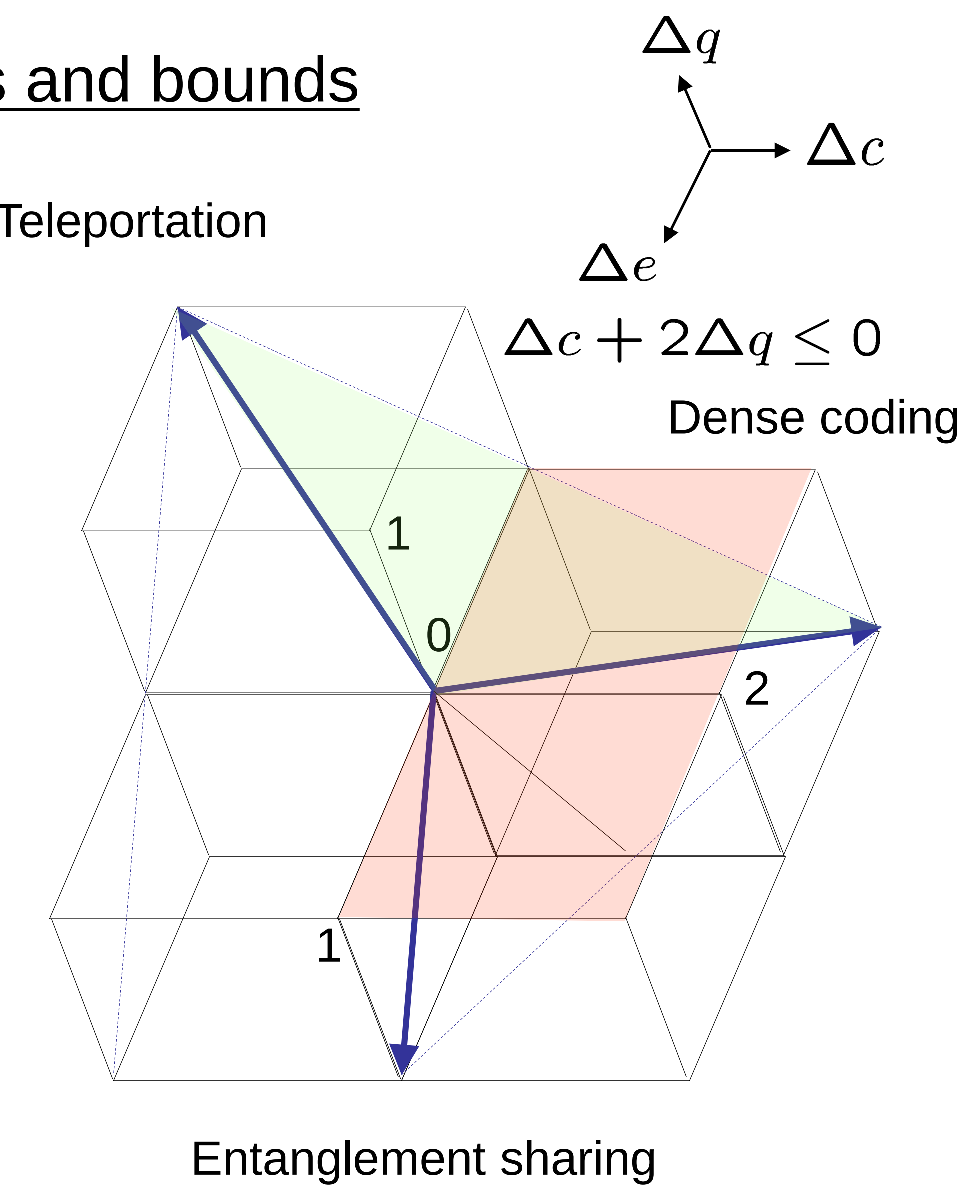
We can do the following…

Restrictions

bits alone ⟶ no ebits

ebits alone ⟶ no bits

1 qubit alone ⟶ no more than 1 bit



Resource conversion protocols and bounds

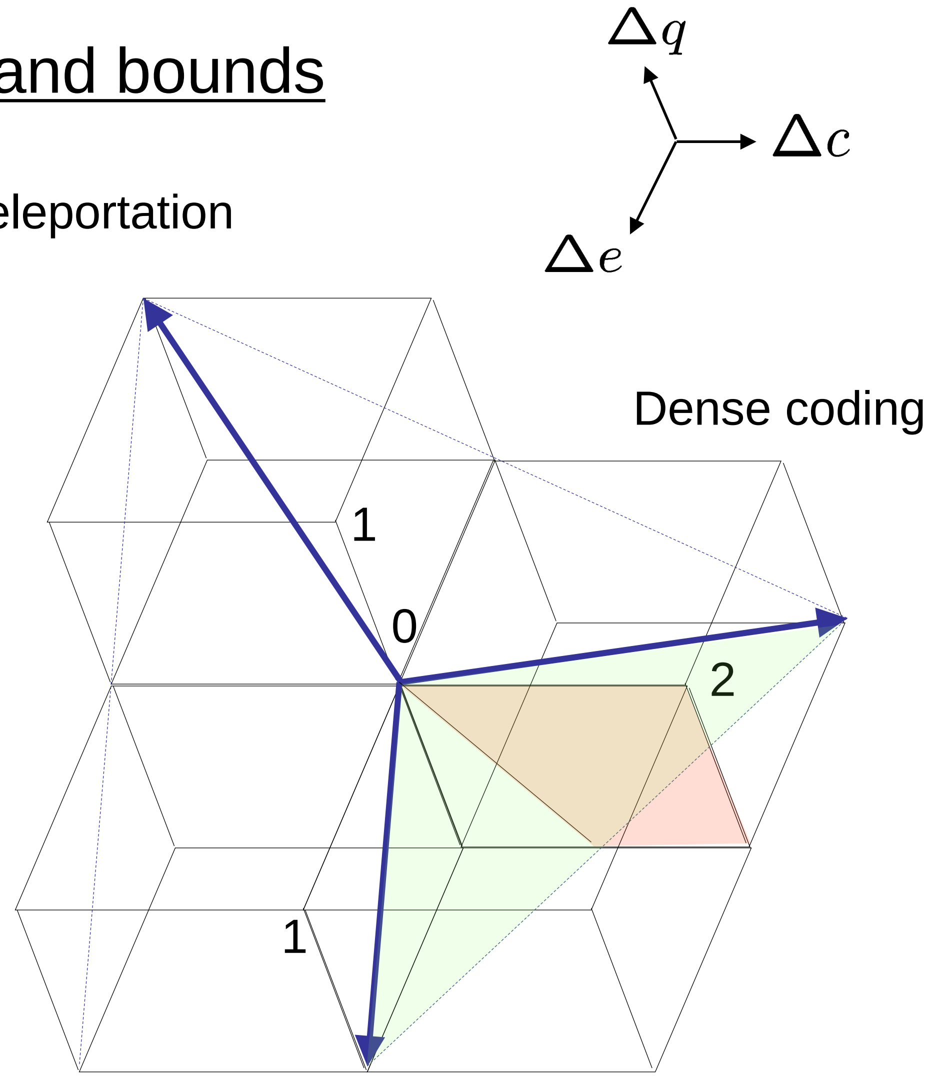
We can do the following…

Restrictions

bits alone $\longrightarrow$ no ebits

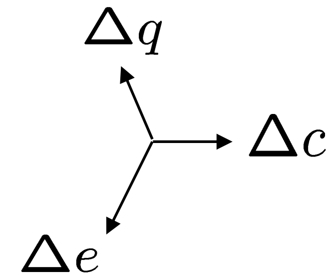
ebits alone $\longrightarrow$ no bits

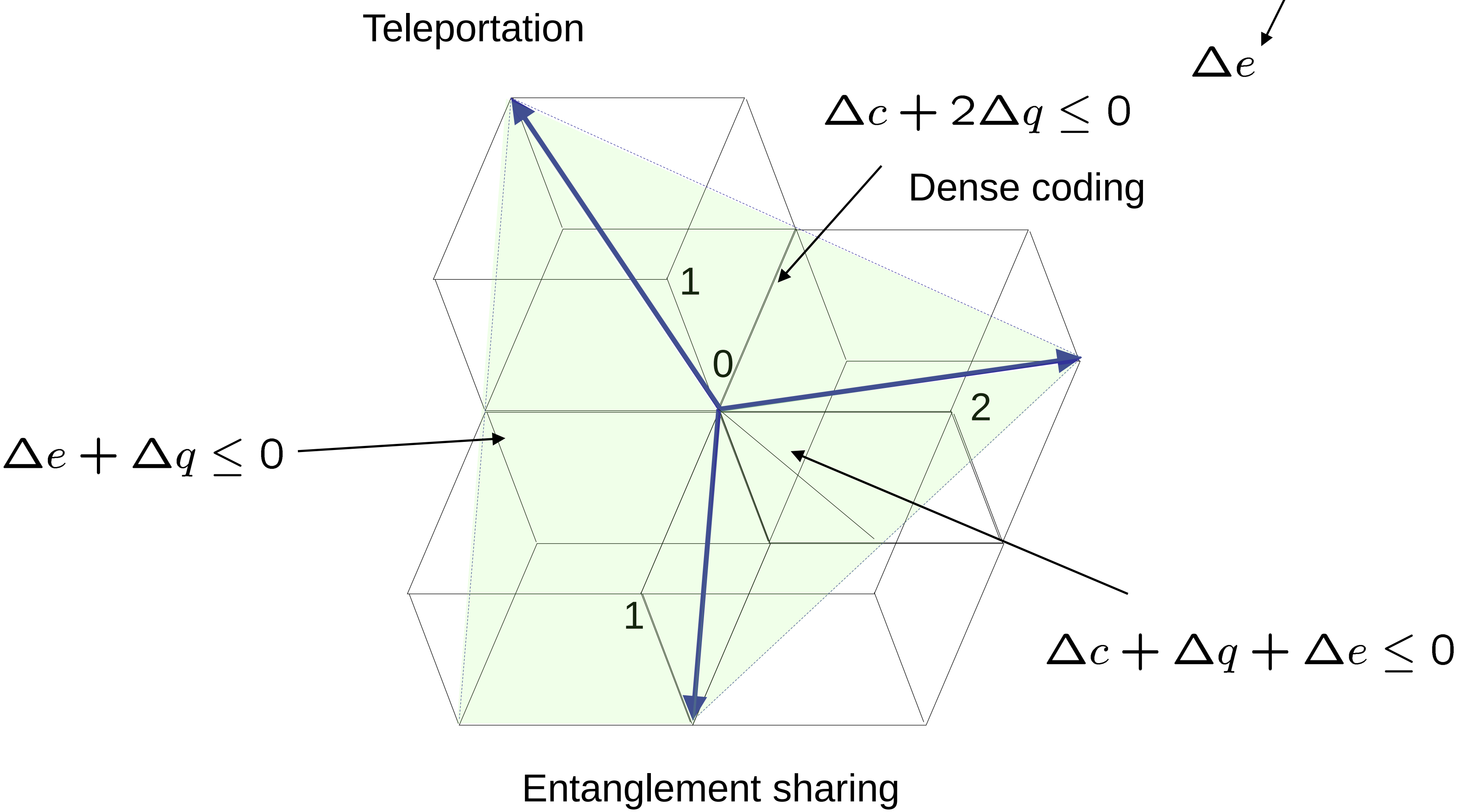
1 qubit alone $\longrightarrow$ no more than 1 bit



$$\Delta c + \Delta q + \Delta e \leq 0$$

Resource conversion protocols and bounds





Resource conversion rule

Entanglement sharing

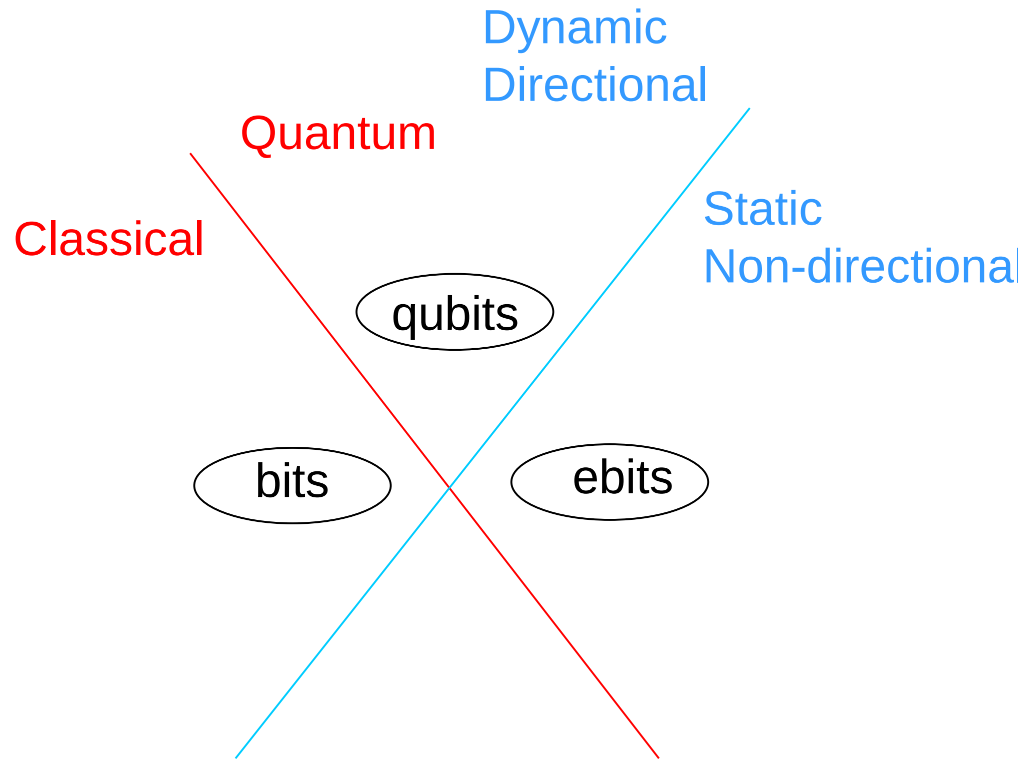
1 qubit $\longrightarrow$ 1 ebit

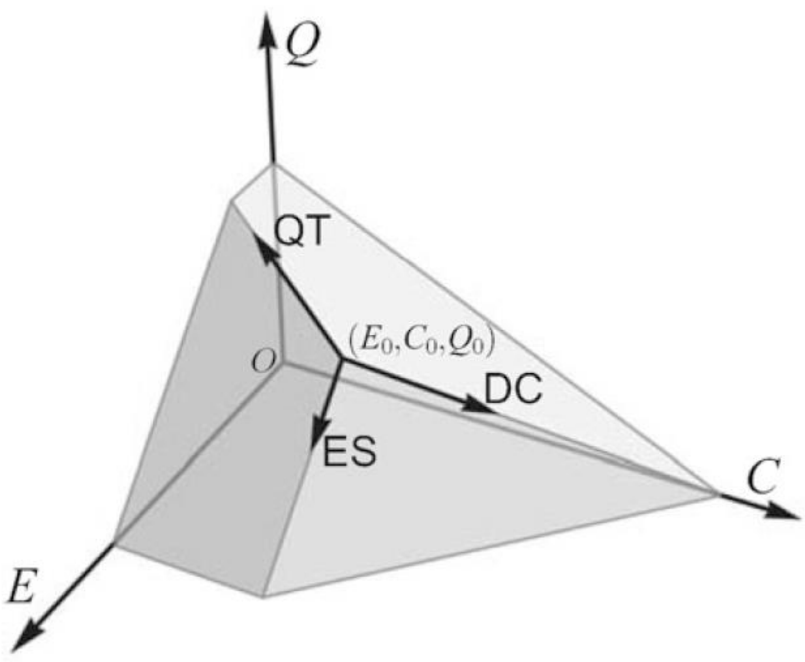
Quantum dense coding

1 qubit + 1 ebit $\longrightarrow$ 2 bits

Quantum teleportation

2 bits + 1 ebit $\longrightarrow$ 1 qubit





Restrictions

bits alone $\longrightarrow$ no ebits

ebits alone $\longrightarrow$ no bits

1 qubit alone $\longrightarrow$ no more than 1 bit

Summary

Basic rules

Vectors
Orthogonal measurements
Unitary transformations

The most general descriptions

Density operators
Generalized measurements
Quantum operations

Important technical tools

Properties of bipartite pure states

- Schmidt decomposition
- Local convertibility
- Relative states
- Bell basis

Measure of distinguishability

Fidelity
No cloning theorem

Communication resources

Classical channels
Quantum channels
Entanglement

Entanglement sharing
Quantum dense coding
Quantum teleportation

Distinction from classical theory

Partially distinguishable pair of pure states

Mixed states are inevitable (entanglement)

Looks as if the state could be chosen retroactively